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Forcing extensions of partial lattices

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Abstract

We prove the following result: *Let K be a lattice, let D be a distributive lattice with zero, and let $\varphi : \text{Con}_c K \rightarrow D$ be a $\{\vee, 0\}$ -homomorphism, where $\text{Con}_c K$ denotes the $\{\vee, 0\}$ -semilattice of all finitely generated congruences of K . Then there are a lattice L , a lattice homomorphism $f : K \rightarrow L$, and an isomorphism $\alpha : \text{Con}_c L \rightarrow D$ such that $\alpha \circ \text{Con}_c f = \varphi$.*

Furthermore, L and f satisfy many additional properties, for example: (i) L is relatively complemented; (ii) L has definable principal congruences; (iii) If the range of φ is cofinal in D , then the convex sublattice of L generated by $f[K]$ equals L .

We mention the following corollaries, that extend many results obtained in the last decades in that area: 1. *Every lattice K such that $\text{Con}_c K$ is a lattice admits a congruence-preserving extension into a relatively complemented lattice.* 2. *Every $\{\vee, 0\}$ -direct limit of a countable sequence of distributive lattices with zero is isomorphic to the semilattice of compact congruences of a relatively complemented lattice with zero.*

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Introduction

Background. The Congruence Lattice Problem (CLP), formulated by R.P. Dilworth in the forties, asks whether every distributive $\{\vee, 0\}$ -semilattice is isomorphic to the semilattice

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$\text{Con}_c L$ of all compact congruences of a lattice L . Despite considerable work in this area, this problem is still open, see [11] for a survey.

In [20], E.T. Schmidt presents an important sufficient condition, for a distributive $\{\vee, 0\}$ -semilattice S , to be isomorphic to the congruence lattice of a lattice. This condition reads “ S is the image of a generalized Boolean lattice under a ‘distributive’ $\{\vee, 0\}$ -homomorphism.” As an important consequence, Schmidt proves the following result.

Theorem 1 (Schmidt, see [21]). *Let S be a distributive lattice with zero. Then there exists a lattice L such that $\text{Con}_c L \cong S$.*

This result is improved in [19], where P. Pudlák proves that one can take L a direct limit of finite atomistic lattices. Although we will not use this fact, we observe that A.P. Huhn proved in [13,14] that Schmidt’s condition is also satisfied by every distributive $\{\vee, 0\}$ -semilattice S such that $|S| \leq \aleph_1$.

The basic statement of CLP can be modified by keeping among the assumptions the distributive $\{\vee, 0\}$ -semilattice S , but by adding to them a *diagram* \mathcal{D} of lattices and a morphism (in the categorical sense) from the image of \mathcal{D} under the Con_c functor to S . (See the end of Section 1 for a precise definition of this functor.) The new problem asks whether one can lift the corresponding *diagram* $\text{Con}_c \mathcal{D} \rightarrow S$ by a *diagram* $\mathcal{D} \rightarrow L$, for some lattice L (that may be restricted to a given class of lattices) and lattice homomorphisms. We cite a few examples.

Theorem 2 (Grätzer and Schmidt, see [10]). *Let K be a lattice. If the lattice $\text{Con } K$ of all congruences of K is finite, then K embeds congruence-preservingly into a sectionally complemented lattice.*

(A lattice L with zero is *sectionally complemented*, if for all $a \leq b$ in L , there exists $x \in L$ such that $a \wedge x = 0$ and $a \vee x = b$.)

Theorem 2 does not extend to the case where $\text{Con } K$ is infinite: by M. Ploščica, J. Tůma, and F. Wehrung [17], the free lattice $\mathbf{F}_L(\omega_2)$ on \aleph_2 generators does not have a congruence-preserving, sectionally complemented extension. In fact, it is proved in J. Tůma and F. Wehrung [24] that $\mathbf{F}_L(\omega_2)$ does not embed congruence-preservingly into a lattice with *permutable* congruences.

Theorem 3 (Grätzer, Lakser, and Wehrung, see [8]; see also Tůma [23]). *Let S be a finite distributive $\{\vee, 0\}$ -semilattice, let \mathcal{D} be a diagram of lattices and lattice homomorphisms consisting of lattices K_0, K_1 , and K_2 , and lattice homomorphisms $f_l: K_0 \rightarrow K_l$, for $l \in \{1, 2\}$. Then any morphism from $\text{Con}_c \mathcal{D}$ to S can be lifted, with respect to the Con_c functor, by a commutative square of lattices and lattice homomorphisms that extends \mathcal{D} .*

The three-dimensional version of Theorem 3, obtained by replacing the truncated square diagram \mathcal{D} by a truncated *cube* diagram, does *not* hold, see J. Tůma and F. Wehrung [24]. On the other hand, the one-dimensional version of Theorem 3 holds, see [6, Theorem 2], or [7, Theorem 4] by G. Grätzer, H. Lakser, and E.T. Schmidt.

As a consequence of Theorem 2, we mention the result that *every distributive $\{\vee, 0\}$ -semilattice of cardinality at most \aleph_1 is isomorphic to the semilattice of compact congruences of a relatively complemented lattice with zero*, see [8]. Hence, lifting results of finite character make it possible to prove representation results of infinite character. The proofs of Theorems 2 and 3 do not extend to infinite S —in fact, we do have a counterexample for the analogue of Theorem 3 for countable S .

In this paper, we prove positive lifting results for *infinite* S , similar to Theorems 2 and 3. The only additional assumption is that S is a *lattice*, just as in [21].

Our first, most general theorem is the following.

Theorem A. *Let D be a distributive lattice with zero, let P be a partial lattice, let $\varphi: \text{Con}_c P \rightarrow D$ be a $\{\vee, 0\}$ -homomorphism. If φ is ‘balanced,’ then it extends to a $\{\vee, 0\}$ -homomorphism $\psi: \text{Con}_c L \rightarrow D$, for a certain lattice L generated, as a lattice, by P .*

We refer to Section 13 for a precise statement of Theorem A. At this point, we observe two facts:

- There are, scattered in the literature, quite a number of nonequivalent definitions of a partial lattice. For example, our definition (see Definition 1.1) is tailored to provide, for a partial lattice P , an *embedding* from $\text{Con } P$ into $\text{Con } \mathbf{F}_L(P)$, where $\mathbf{F}_L(P)$ denotes the free lattice on P . It is *not* equivalent to the definition presented in [5].
- The condition that φ be ‘balanced’ (see Definition 13.3) is quite complicated, which explains to a large extent the size of this paper.

Intuitively, the condition that φ be balanced means that the computation of finitely generated *ideals* and *filters*, as well as finite intersections and joins of these, in every quotient of P by a prime ideal G of D , can be captured by *finite* amounts of information, and this uniformly on G . This condition is so difficult to formulate that it appears at first sight as quite unpractical.

However, it is satisfied in two important cases, namely: either P is a *lattice* (and then the statement of Theorem A trivializes, as it should), or P is *finite* with nonempty domains for the meet and the join, see Proposition 12.7. Although this observation is quite easy, the next one is far less trivial. It shows that a large amount of *amalgams* of balanced partial lattices and homomorphisms are balanced, see Proposition 18.5.

Theorem B. *Let D be a distributive lattice with zero. Let K be a finite lattice, let P and Q be partial lattices each of them is either a finite partial lattice or a lattice, let $f: K \rightarrow P$ and $g: K \rightarrow Q$ be homomorphisms of partial lattices, let $\mu: \text{Con}_c P \rightarrow D$ and $\nu: \text{Con}_c Q \rightarrow D$ such that $\mu \circ \text{Con}_c f = \nu \circ \text{Con}_c g$. Then there exist a lattice L , homomorphisms of partial lattices $\tilde{f}: P \rightarrow L$ and $\tilde{g}: Q \rightarrow L$, and a $\{\vee, 0\}$ -homomorphism $\varphi: \text{Con}_c L \rightarrow D$ such that $\tilde{f} \circ f = \tilde{g} \circ g$, $\mu = \varphi \circ \text{Con}_c \tilde{f}$, and $\nu = \varphi \circ \text{Con}_c \tilde{g}$. Furthermore, the construction can be done in such a way that the following additional properties hold:*

- (i) L is generated, as a lattice, by $\tilde{f}[P] \cup \tilde{g}[Q]$.
- (ii) The map φ isolates 0.

(We say that a map φ isolates 0, if $\varphi(\theta) = 0$ iff $\theta = 0$, for all θ in the domain of φ .)

Unlike what happens with Theorem A, stating Theorem B does not require any complicated machinery—it is an immediately usable tool.

We can now state our one-dimensional lifting result.

Theorem C. *Let K be a lattice, let D be a distributive lattice with zero, and let $\varphi: \text{Con}_c K \rightarrow D$ be a $\{\vee, 0\}$ -homomorphism. There are a relatively complemented lattice L of cardinality $|K| + |D| + \aleph_0$, a lattice homomorphism $f: K \rightarrow L$, and an isomorphism $\alpha: \text{Con}_c L \rightarrow D$ such that the following assertions hold:*

- (i) $\varphi = \alpha \circ \text{Con}_c f$.
- (ii) The range of f is coinitial (respectively cofinal) in L .
- (iii) If the range of φ is cofinal in D , then the range of f is internal in L .

We observe that for a distributive semilattice D with zero, Theorem C characterizes D being a lattice, see [25].

Here, we say that a subset X of a lattice L is *coinitial* (*cofinal*, *internal*, respectively) if the upper subset (lower subset, convex subset, respectively) generated by X equals L .

The information that L be relatively complemented in the statement of Theorem C reflects only part of the truth. It turns out that L satisfies certain strong closure conditions—we say that $\langle L, \alpha \rangle$ is *internally saturated*, see Definition 19.2. This statement implies the following properties of L , see Proposition 20.8 for details:

- (i) L is relatively complemented.
- (ii) L has *definable principal congruences*. More precisely, there exists a positive existential formula $\Phi(x, y, u, v)$ of the language of lattice theory such that for every internally saturated $\langle L, \alpha \rangle$ and all $a, b, c, d \in L$,

$$\Theta_L(a, b) \subseteq \Theta_L(c, d) \quad \text{iff} \quad L \text{ satisfies } \Phi(a, b, c, d).$$

A similar result is easily seen to hold for statements of the form $\Theta_L(a, b) \subseteq \bigvee_{i < n} \Theta_L(c_i, d_i)$.

Then Theorems B and C together imply easily the following two-dimensional lifting result, that widely extends the main result of G. Grätzer, H. Lakser, and F. Wehrung [8].

Theorem D. *Let K, P, Q, f, g, μ, ν satisfy the assumptions of Theorem B. Then there are a relatively complemented lattice L of cardinality $|P| + |Q| + |D| + \aleph_0$, homomorphisms of partial lattices $\bar{f}: P \rightarrow L$ and $\bar{g}: Q \rightarrow L$, and an isomorphism $\varphi: \text{Con}_c L \rightarrow D$ such that $\bar{f} \circ f = \bar{g} \circ g$, $\mu = \varphi \circ \text{Con}_c f$, and $\nu = \varphi \circ \text{Con}_c g$. Furthermore, the construction can be done in such a way that the following additional properties hold:*

- (i) The subset $\bar{f}[P] \cup \bar{g}[Q]$ generates L as an ideal (respectively filter).
- (ii) If the subsemilattice of D generated by $\mu[\text{Con}_c P] \cup \nu[\text{Con}_c Q]$ is cofinal in D , then $\bar{f}[P] \cup \bar{g}[Q]$ generates L as a convex sublattice.

Furthermore, once Theorem C is proved, easy corollaries follow. For example,

Corollary 21.1. *Every lattice K such that $\text{Con}_c K$ is a lattice has an internal, congruence-preserving embedding into a relatively complemented lattice.*

Corollary 21.3. *Every $\{\vee, 0\}$ -semilattice that is a direct limit of a countable sequence of distributive lattices with zero is isomorphic to the semilattice of compact congruences of a relatively complemented lattice with zero.*

Methods. Our methods of proof, especially for Theorems A and B, are radically different from the usual ‘finite’ methods, for example, those used in the proofs of Theorems 2 and 3. In some sense, we take the most naive possible approach of the problem. We are given a partial lattice P , a distributive lattice D with zero, a homomorphism $\varphi : \text{Con}_c P \rightarrow D$, and we wish to “extend P to a relatively complemented lattice L , and make φ an isomorphism,” as in the statement of Theorem A. So we “add new joins and meets” in order to make P a *total* lattice (we use Theorem A), we “add relative complements” in order to make P relatively complemented (see Lemma 20.1), we “force projectivity of intervals” in order to make φ an embedding (see Lemmas 20.3–20.6), and we “add new intervals” in order to make φ surjective (see Lemma 20.7). Of course, the main problem is then to confine the range of φ within D .

In this sense, this approach is related to G. Grätzer and E.T. Schmidt’s [9] proof of the representation problem of congruence lattices of algebras, see also P. Pudlák [18] and E.T. Schmidt [22, Section 2.3]: given an algebraic (not necessarily distributive) lattice A , a partial algebra U constructed so that $\text{Con } U \cong A$, then U is extended to a total algebra with the same congruence lattice.

However, there is an important difference between this approach and ours, namely: in Grätzer and Schmidt’s proof, infinitely many new operations need to be incorporated to the signature of the algebra. This restriction is absolutely unavoidable, as proves W. Lampe’s result (a stronger version was proved independently by R. Freese and W. Taylor) that certain algebraic lattices require many operations to be represented, see R. Freese, W. Lampe, and W. Taylor [4], or E.T. Schmidt [22, Section 2.4]. In the present paper, we are restricted to the language of lattice theory $\langle \vee, \wedge \rangle$. This may partly explain our restriction to algebraic lattices which are ideal lattices of distributive *lattices*. That the latter restriction is necessary is established in the forthcoming paper J. Tůma and F. Wehrung [25].

To get around this difficulty, we borrow the notations and methods of the theory of forcing and Boolean-valued models. Although it has been recognized that the latter are, in universal algebra, a more convenient framework than the usual sheaf representation results, see, for example, in S. Burris and H.P. Sankappanavar [1, Chapter IV], one can probably not say that they are, at the present time, tools of common use in lattice theory. For this reason, our presentation will assume no familiarity with Boolean-valued models. We refer the reader, for example, to T. Jech [16] for a presentation of this topic.

The basic idea of the present paper is, actually, quite simple. For a partial lattice P , we consider the standard construction of the free lattice $\mathbf{F}_L(P)$ on P . More specifically, $\mathbf{F}_L(P)$ is constructed as the set of *words* on P , using the binary operations \vee and \wedge . The *ordering* on $\mathbf{F}_L(P)$ is defined inductively, see Definition 2.6. This can be done by assigning to every

statement of the form $\dot{x} \leq \dot{y}$, where \dot{x} and \dot{y} are words on P , a ‘truth value’ $\|\dot{x} \leq \dot{y}\|$, equal either to 0 (false) or 1 (true). So $\|\dot{x} \leq \dot{y}\|$ equals 1 if $\dot{x} \leq \dot{y}$, 0 otherwise. So, for example, rule (ii) of Definition 2.6 may be stated as

$$\|\dot{x}_0 \vee \dot{x}_1 \leq \dot{y}_0 \wedge \dot{y}_1\| = \bigwedge_{i,j < 2} \|\dot{x}_i \leq \dot{y}_j\|. \quad (0.1)$$

If the truth values of statements are no longer confined to $\{0, 1\}$ but, rather, to elements of a given distributive lattice D (which has to be thought as the *dual* lattice of the lattice D of the statements of Theorems A and B), (0.1) becomes part of the inductive definition of a map that with every pair $\langle \dot{x}, \dot{y} \rangle$ of words on P associates the ‘truth value’ $\|\dot{x} \leq \dot{y}\| \in D$, that we shall still call “Boolean value” (after all, D embeds into a Boolean algebra).

In this way, it seems at first sight a trivial task to extend Definition 2.6 to a D -valued context. However, the major obstacle remains of the computation of $\|\dot{x} \leq \dot{y}\|$ at the *bottom* level, that is, for \dot{x} and \dot{y} finite meets or joins of elements of P . This situation is not unlike what happens in set theory, where the main problem in defining Boolean values in the Scott–Solovay Boolean universe is to define them on the *atomic* formulas, see [15]. In fact, the method used in [26] reflects more closely what is done in the present paper, namely, the domain of the Boolean value function is extended from a set of ‘urelements’ to the universe of set theory that they generate.

The condition that $\langle P, \varphi \rangle$ be balanced is designed to ensure that these Boolean values belong to D , while they would typically, in the general case, belong to the completion of the universal Boolean algebra of D .

The reader may feel at this point a slight uneasiness, because the distributive lattice D in which the Boolean values live is related to the *dual* of $\text{Conc } P$, rather than to $\text{Conc } P$ itself. It seems, indeed, pointless to dualize D , prove a large amount of results on the dual, and then dualize again to recover D . Why bother doing this? The alternative would be to stick with the original D , and so, to interpret the Boolean values by $\|\dot{x} \leq \dot{x}\| = 0$ (instead of 1, ‘true’), and $\|\dot{x} \leq \dot{z}\| \leq \|\dot{x} \leq \dot{y}\| \vee \|\dot{y} \leq \dot{z}\|$ (instead of the dual, see Definition 4.1(ii)). Furthermore, one would have to interpret the propositional connective ‘and’ by the join \vee , and ‘or’ by the meet \wedge , and so on. This is definitely unattractive to the reader familiar with Boolean models. Of course, the last decisive argument for one way or the other is merely related to a matter of taste.

We now give a short summary of the paper, part by part.

Part 1 introduces partial lattices and their congruences, and also the free lattice on a partial lattice. A noticeable difference between our definition of a congruence and the usual definition of a congruence is that our congruences are *not symmetric* in general. The reason for this is very simple, namely, if $f : P \rightarrow Q$ is a homomorphism of partial lattices, then its *kernel*, instead of being defined as usual as the set of all pairs $\langle x, y \rangle$ such that $f(x) = f(y)$, is defined here as the set of all $\langle x, y \rangle$ such that $f(x) \leq f(y)$. Of course, for (total) lattices, the two resulting definitions of a congruence are essentially equivalent—in particular, they give isomorphic congruence lattices.

In Section 3, we interpret the classical operation of ‘pasting’ two partial lattices above a lattice as a *pushout* in the category of partial lattices and homomorphisms of partial

lattices. Although the description of the pushout, Proposition 3.4, is fairly straightforward, it paves the way for its D -valued analogue, Proposition 15.4.

Part 2 begins with the simple definitions, in Section 4, of a D -valued poset or of a D -valued partial lattice. The purpose of Section 6 is to introduce the important definition of a *sample*, that makes it possible, *via* additional assumptions, to extend to the D -valued world the classical notions of *ideal* and *filter* of a partial lattice P , see Section 2.1. The corresponding D -valued notions, instead of corresponding to *subsets* of P , correspond to *functions* from P to D .

However, the objects we wish to solve problems about are not D -valued partial lattices, but plain partial lattices. Thus we present in **Part 3** a class of structures that live simultaneously in both worlds, the *D -comeasured partial lattices*, to which we translate the results of Part 2. In order to extend to a *lattice* the Boolean values defined on the original partial lattice, we introduce the definition of a *balanced D -comeasured partial lattice*, see Definition 13.3. Then we prove, in Sections 16 and 17, that all our finiteness conditions (they add up to the condition of being balanced) are preserved under amalgamation above a finite lattice.

Now that all this hard technical work is completed, we start applying it in **Part 4**. Most arguments used in this part are based on simple amalgamation constructions of partial lattices above finite lattices, that all yield, by our previous work, balanced D -comeasured partial lattices.

Notation and terminology

For a set X , we denote by $[X]^{<\omega}$ (respectively $[X]_*^{<\omega}$) the set of all finite (respectively nonempty finite) subsets of X .

We put $\mathbf{2} = \{0, 1\}$, endowed with its canonical structure of lattice. For a nonnegative integer n , we identify n with $\{0, 1, \dots, n-1\}$.

Let P be a preordered set. For subsets X, Y of P , let $X \leq Y$ be the statement $\forall x \in X, \forall y \in Y, x \leq y$. We shall write $a \leq X$ (respectively $X \leq a$) instead of $\{a\} \leq X$ (respectively $X \leq \{a\}$). A subset X of P is a *lower subset* (respectively *upper subset*) of P if for all $x \leq y$ in P , $y \in X$ (respectively $x \in X$) implies that $x \in X$ (respectively $y \in X$). We say that X is a *convex subset* of P , if $a \leq x \leq b$ and $\{a, b\} \subseteq X$ implies that $x \in X$, for all $a, b, x \in P$.

If $X \subseteq P$, we denote by $\downarrow X$ (respectively $\uparrow X$) the lower subset (respectively upper subset) of P generated by X . For $a \in P$, we put $\downarrow a = \downarrow \{a\}$ and $\uparrow a = \uparrow \{a\}$.

For $a \in P$ and $X \subseteq P$, let $a = \sup X$ be the statement

$$X \leq a \quad \text{and} \quad \forall x, X \leq x \Rightarrow a \leq x.$$

The statement $a = \inf X$ is defined dually. Note that if $a = \sup X$, then $a' = \sup X$ for all a' equivalent to a with respect to the preordering \leq (that is, $a \leq a' \leq a$).

For a preordering α of a set P and for $x, y \in P$, the statement $\langle x, y \rangle \in \alpha$ will often be abbreviated $x \leq_\alpha y$.

For a lattice L , L^d denotes the *dual lattice* of L .

Part 1. Partial lattices

1. Partial prelattices and partial lattices

Definition 1.1. A *partial prelattice* is a structure $\langle P, \leq, \bigvee, \bigwedge \rangle$, where P is a nonempty set, \leq is a preordering on P , and \bigvee, \bigwedge are partial functions from $[P]_*^{<\omega}$ to P satisfying the following properties:

- (i) $a = \bigvee X$ implies that $a = \sup X$, for all $a \in P$ and all $X \in [P]_*^{<\omega}$;
- (ii) $a = \bigwedge X$ implies that $a = \inf X$, for all $a \in P$ and all $X \in [P]_*^{<\omega}$.

We say that P is a *partial lattice*, if \leq is antisymmetric.

A *congruence* of P is a preordering \preceq of P containing \leq such that $\langle P, \preceq, \bigvee, \bigwedge \rangle$ is a partial prelattice.

If P and Q are partial prelattices, a *homomorphism of partial prelattices* from P to Q is an order-preserving map $f: P \rightarrow Q$ such that $a = \bigvee X$ (respectively $a = \bigwedge X$) implies that $f(a) = \bigvee f[X]$ (respectively $f(a) = \bigwedge f[X]$), for all $a \in P$ and all $X \in [P]_*^{<\omega}$. We say that a homomorphism f is an *embedding*, if $f(a) \leq f(b)$ implies that $a \leq b$, for all $a, b \in P$.

We shall naturally identify *lattices* with partial lattices P such that \bigvee and \bigwedge are defined everywhere on $[P]_*^{<\omega}$.

Remark 1.2. For an embedding $f: P \rightarrow Q$ of partial lattices, we do *not* require that $\bigvee f[X]$ be defined implies that $\bigvee X$ is defined (and dually), for $X \in [P]_*^{<\omega}$.

Proposition 1.3. Let P be a partial prelattice. Then the set $\text{Con } P$ of all congruences of P is a closure system in the powerset lattice of $P \times P$, closed under directed unions. In particular, it is an algebraic lattice.

We denote by $\text{Con}_c P$ the $\{\vee, 0\}$ -semilattice of all *compact* congruences of P , by $\mathbf{0}_P$ the least congruence of P (that is, $\mathbf{0}_P$ is the preordering of P), and by $\mathbf{1}_P$ the largest (coarse) congruence of P . The map $P \mapsto \text{Con}_c P$ can be extended in a natural way in a *functor*, as follows. For a homomorphism $f: P \rightarrow Q$ of partial lattices, we define a $\{\vee, 0\}$ -homomorphism $\text{Con}_c f: \text{Con}_c P \rightarrow \text{Con}_c Q$ as the map that with every congruence α of P associates the congruence of Q generated by all pairs $\langle f(x), f(y) \rangle$, for $\langle x, y \rangle \in \alpha$.

If P is a lattice, then $\text{Con } P$ is distributive, but this may not hold for a general partial lattice P .

For $a, b \in P$, we denote by $\Theta_P^+(a, b)$ the least congruence θ of P such that $a \leq_\theta b$, and we put $\Theta_P(a, b) = \Theta_P^+(a, b) \vee \Theta_P^+(b, a)$, the least congruence θ of P such that $a \equiv_\theta b$. Of course, the congruences of the form $\Theta_P^+(a, b)$ are generators of the join-semilattice $\text{Con}_c P$.

2. The free lattice on a partial lattice

We present in this section an explicit construction, due to R.A. Dean [2], of the free lattice on a partial lattice, see also [3, p. 249]. For the needs of this paper, the definitions are slightly modified (in particular, the relation \preccurlyeq defined below, see Definition 2.6), but it is easy to verify that they are, in fact, equivalent to the original ones.

Throughout this section, we shall fix a partial lattice P .

2.1. Ideals, filters

Definition 2.1. An *ideal* of P is a lower subset I of P such that $X \subseteq I$ and $a = \bigvee X$ imply that $a \in I$, for all $X \in [P]_*^{\leq \omega}$ and all $a \in P$. Dually, a *filter* of P is an upper subset F of P such that $X \subseteq F$ and $a = \bigwedge X$ imply that $a \in F$, for all $X \in [P]_*^{\leq \omega}$ and all $a \in P$.

We observe that both \emptyset and P are simultaneously an ideal and a filter of P . For $a \in P$, $\downarrow a$ is an ideal of P (*principal ideal*), while $\uparrow a$ is a filter of P (*principal filter*). In case P is a *lattice* (that is, \bigvee and \bigwedge are everywhere defined), the ideals of the form $\downarrow a$ are the only nonempty finitely generated ideals of P .

Lemma 2.2. The set $\mathcal{I}(P)$ (respectively $\mathcal{F}(P)$) of all ideals (respectively filters) of P is a closure system in the powerset lattice $\mathcal{P}(P)$ of P , closed under directed unions. Hence, both $\mathcal{I}(P)$ and $\mathcal{F}(P)$ are algebraic lattices.

2.2. Description of the free lattice on P

Notation 2.3. For a set Ω , let $W(\Omega)$ denote the set of *terms* on Ω and the two binary operations \vee and \wedge .

So, the elements of $W(\Omega)$ are formal “polynomials” on the elements of Ω , such as $((a \vee b) \vee c) \wedge (d \vee e)$, where $a, b, c, d, e \in \Omega$, etc. The *height* of an element \dot{x} of $W(\Omega)$ is defined inductively by $\text{ht}(a) = 0$ for $a \in \Omega$, and $\text{ht}(\dot{x} \wedge \dot{y}) = \text{ht}(\dot{x} \vee \dot{y}) = \text{ht}(\dot{x}) + \text{ht}(\dot{y}) + 1$.

We shall now specialize to the case where Ω is the underlying set of the partial lattice P . (In Section 11, the notation $W(P)$ will be used for structures P that are not necessarily partial lattices.)

Definition 2.4. For $\dot{x} \in W(P)$, we define, by induction on the height $\text{ht}(\dot{x})$ of \dot{x} , an ideal \dot{x}^- of P and a filter \dot{x}^+ of P as follows:

- (i) $\dot{x}^- = \downarrow a$ and $\dot{x}^+ = \uparrow a$, if $\dot{x} = a \in P$.
- (ii) If $\dot{x} = \dot{x}_0 \vee \dot{x}_1$, we put $\dot{x}^- = \dot{x}_0^- \vee \dot{x}_1^-$ (the join being computed in $\mathcal{I}(P)$), and $\dot{x}^+ = \dot{x}_0^+ \cap \dot{x}_1^+$.
- (iii) If $\dot{x} = \dot{x}_0 \wedge \dot{x}_1$, we put $\dot{x}^- = \dot{x}_0^- \cap \dot{x}_1^-$, and $\dot{x}^+ = \dot{x}_0^+ \vee \dot{x}_1^+$ (the join being computed in $\mathcal{F}(P)$).

Definition 2.5. For $\dot{x}, \dot{y} \in W(P)$, we define $\dot{x} \ll \dot{y}$ to hold, if $\dot{x}^+ \cap \dot{y}^- \neq \emptyset$.

Definition 2.6. We define inductively a binary relation \preceq on $\mathbf{W}(P)$, as follows:

- (i) $\dot{x} \preceq \dot{y}$ iff $\dot{x} \ll \dot{y}$, for all $\dot{x}, \dot{y} \in \mathbf{W}(P)$ such that $\dot{x} \in P$ or $\dot{y} \in P$.
- (ii) $\dot{x}_0 \vee \dot{x}_1 \preceq \dot{y}_0 \wedge \dot{y}_1$ iff $\dot{x}_i \preceq \dot{y}_j$, for all $i, j < 2$.
- (iii) $\dot{x}_0 \vee \dot{x}_1 \preceq \dot{y}_0 \vee \dot{y}_1$ iff $\dot{x}_i \preceq \dot{y}_0 \vee \dot{y}_1$, for all $i < 2$.
- (iv) $\dot{x}_0 \wedge \dot{x}_1 \preceq \dot{y}_0 \wedge \dot{y}_1$ iff $\dot{x}_0 \wedge \dot{x}_1 \preceq \dot{y}_j$, for all $j < 2$.
- (v) $\dot{x}_0 \wedge \dot{x}_1 \preceq \dot{y}_0 \vee \dot{y}_1$ iff either $\dot{x}_0 \wedge \dot{x}_1 \ll \dot{y}_0 \vee \dot{y}_1$ or $\dot{x}_i \preceq \dot{y}_j$, for some $i, j < 2$.

The relevant observations can be summarized in the following form.

Lemma 2.7. Let $a, b \in P$ and let $\dot{x}, \dot{y}, \dot{z} \in \mathbf{W}(P)$. Then the following assertions hold:

- (i) $a \in \dot{x}^-$ and $b \in \dot{x}^+$ imply that $a \leq b$;
- (ii) $a \in \dot{x}^-$ and $\dot{x} \preceq \dot{y}$ imply that $a \in \dot{y}^-$;
- (iii) $a \in \dot{y}^+$ and $\dot{x} \preceq \dot{y}$ imply that $a \in \dot{x}^+$;
- (iv) $\dot{x} \ll \dot{y}$ implies that $\dot{x} \preceq \dot{y}$;
- (v) $\dot{x} \preceq \dot{x}$;
- (vi) $\dot{x} \preceq \dot{y}$ and $\dot{y} \preceq \dot{z}$ imply that $\dot{x} \preceq \dot{z}$.

Let \equiv denote the equivalence relation associated with the preordering \preceq . We define $\mathbf{F}_L(P) = \langle \mathbf{W}(P), \preceq \rangle / \equiv$. Let $j_P : P \rightarrow \mathbf{F}_L(P)$, the *natural map*, be defined by $j_P(a) = a / \equiv$, for all $a \in P$.

Proposition 2.8. The poset $\mathbf{F}_L(P)$ is a lattice and j_P is an embedding of partial lattices. Furthermore, j_P is universal among all the homomorphisms of partial lattices from P to a lattice.

So we identify $\mathbf{F}_L(P)$ (together with the natural map j_P) with the *free lattice on the partial lattice P* , that is, the lattice defined by generators \check{a} ($a \in P$) and relations $\check{a} = \check{x}_0 \vee \cdots \vee \check{x}_{n-1}$ (respectively $\check{a} = \check{x}_0 \wedge \cdots \wedge \check{x}_{n-1}$) if $a = \bigvee \{x_0, \dots, x_{n-1}\}$ (respectively $a = \bigwedge \{x_0, \dots, x_{n-1}\}$) in P .

2.3. Generation of ideals and filters

In any lattice, the finitely generated ideals are exactly the principal ideals, and similarly for filters. In general partial lattices, the situation is much more complicated. The somewhat more precise description of ideals and filters that we shall give in this section will be used later in Section 7.

Definition 2.9. Let X and U be subsets of P . For $n < \omega$, we define, by induction on n , a subset $\text{Id}_n(X, U)$ of P , as follows:

- (i) $\text{Id}_0(X, U) = \downarrow X$.
- (ii) $\text{Id}_{n+1}(X, U)$ is the union of $\text{Id}_n(X, U)$ and the lower subset of P generated by all elements of the form $\bigvee Z$, where $\emptyset \subset Z \subseteq U \cap \text{Id}_n(X, U)$ and $\bigvee Z$ is defined (\subset denotes *proper* inclusion).

Dually, we define, by induction on n , a subset $\text{Fil}_n(X, U)$ of P , as follows:

- (i) $\text{Fil}_0(X, U) = \uparrow X$.
- (ii) $\text{Fil}_{n+1}(X, U)$ is the union of $\text{Fil}_n(X, U)$ and the upper subset of P generated by all elements of the form $\bigwedge Z$, where $\emptyset \subset Z \subseteq U \cap \text{Fil}_n(X, U)$ and $\bigwedge Z$ is defined.

We observe, in particular, that $\bigcup_{n < \omega} \text{Id}_n(X, P)$ is the ideal $\text{Id}(X)$ of P generated by X . The subsets $\text{Id}_n(X, U)$, for finite U , can be viewed as “finitely generated approximations” of $\text{Id}(X)$. Similar considerations hold for $\text{Fil}_n(X, U)$ and $\text{Fil}(X) = \bigcup_{n < \omega} \text{Fil}_n(X, P)$.

3. Amalgamation of partial lattices above a lattice

Most of the results of this section are folklore, we recall them here for convenience.

Definition 3.1. A *V-formation* of partial lattices is a structure $\langle K, P, Q, f, g \rangle$ subject to the following conditions:

- (V1) K, P, Q are partial lattices.
- (V2) $f: K \hookrightarrow P$ and $g: K \hookrightarrow Q$ are embeddings of partial lattices.

A V-formation $\langle K, P, Q, f, g \rangle$ is *standard*, if the following conditions hold:

- (SV1) K is a lattice.
- (SV2) $K = P \cap Q$ (set-theoretically), and f and g are, respectively, the inclusion map from K into P and the inclusion map from K into Q .

Of course, we identify a V-formation $\langle K, P, Q, f, g \rangle$ with the *diagram* of partial lattices that consists of two arrows from K , one of them $f: K \rightarrow P$, the other $g: K \rightarrow Q$.

Furthermore, the homomorphisms in standard V-formations are understood (they are the inclusion maps), so, in that case, we shall write $\langle K, P, Q \rangle$ instead of $\langle K, P, Q, f, g \rangle$.

The following lemma is a set-theoretical triviality.

Lemma 3.2. Every V-formation $\langle K, P, Q, f, g \rangle$ of partial lattices, with K a lattice, is isomorphic to a standard V-formation.

Definition 3.3. Let $\mathcal{D} = \langle K, P, Q, f, g \rangle$ be a V-formation of partial lattices. An *amalgam* of \mathcal{D} is a triple $\langle R, f', g' \rangle$, where R is a partial lattice and $f': P \hookrightarrow R$, $g': Q \hookrightarrow R$ are embeddings of partial lattices such that $f' \circ f = g' \circ g$.

As usual in category-theoretical terminology, we say that a *pushout* of \mathcal{D} is any initial object in the category of amalgams of \mathcal{D} with their *homomorphisms* (not only embeddings). Of course, if the pushout of \mathcal{D} exists, then it is unique up to isomorphism.

We shall be concerned about not only the existence but also the *description* of pushouts in a very precise context:

Proposition 3.4. *Let $\mathcal{D} = \langle K, P, Q, f, g \rangle$ be a V-formation of partial lattices, with K a lattice. Then \mathcal{D} has a pushout. Furthermore, assume that \mathcal{D} is a standard V-formation. Then the pushout $\langle R, f', g' \rangle$ of \mathcal{D} can be described by the following data:*

- (a) $R = P \cup Q$, endowed with the partial ordering \leq consisting of all pairs $\langle x, y \rangle$ of elements of R satisfying the following conditions:
 - (a1) $x, y \in P$ and $x \leq_P y$;
 - (a2) $x, y \in Q$ and $x \leq_Q y$;
 - (a3) $x \in P, y \in Q$, and there exists $z \in K$ such that $x \leq_P z$ and $z \leq_Q y$;
 - (a4) $x \in Q, y \in P$, and there exists $z \in K$ such that $x \leq_Q z$ and $z \leq_P y$.
- (b) For $a \in R$ and $X \in [R]_*^{\leq \omega}$, $a = \bigvee X$ holds in R iff either $X \cup \{a\} \subseteq P$ and $a = \bigvee X$ in P or $X \cup \{a\} \subseteq Q$ and $a = \bigvee X$ in Q .
- (b*) For $a \in R$ and $X \in [R]_*^{\leq \omega}$, $a = \bigwedge X$ holds in R iff either $X \cup \{a\} \subseteq P$ and $a = \bigwedge X$ in P or $X \cup \{a\} \subseteq Q$ and $a = \bigwedge X$ in Q .
- (c) f' (respectively g') is the inclusion map from P into R (respectively from Q into R).

Note. It is easy to prove that any diagram of partial lattices admits a colimit. In particular, pushouts always exist. However, we are, in Proposition 3.4, more interested in the *description* of the pushout.

Proof. The fact that the binary relation \leq defined above on R is a partial ordering is folklore (and easy to verify).

Now we prove that R is a partial lattice. We first observe that since K is a partial sublattice of both P and Q , the partial operations \bigvee and \bigwedge on R described in (b) and (b*) above are, indeed, partial functions.

Let $\langle a, X \rangle \in R \times [R]_*^{\leq \omega}$ such that $a = \bigvee X$ in R , we prove that $a = \sup X$ in R . By the definition of \bigvee in R , $a = \bigvee X$ holds either in P or in Q , so, without loss of generality, $X \cup \{a\} \subseteq P$ and $a = \bigvee X$ in P . Since P is a partial lattice, it follows that

$$a = \sup X \quad \text{in } P. \quad (3.1)$$

From $X \leq_P a$ follows that $X \leq a$. Now let $b \in R$ such that $X \leq b$, we prove that $a \leq b$. If $b \in P$, then $X \leq_P b$, thus, by (3.1), $a \leq_P b$, so $a \leq b$.

Now suppose that $b \in Q$. For all $x \in X$, $x \leq b$ with $x \in P$ and $b \in Q$, thus there exists $x^* \in K$ such that

$$x \leq_P x^*, \quad (3.2)$$

$$x^* \leq_Q b. \quad (3.3)$$

Since K is a lattice, $c = \bigvee_{x \in X} x^*$ is defined in K . Since K is a partial sublattice of Q , the equality $c = \bigvee_{x \in X} x^*$ also holds in Q . Thus, by (3.3), we obtain the inequality

$$c \leq_Q b. \quad (3.4)$$

Furthermore, for $x \in X$, $x^* \leq_K c$, thus $x^* \leq_P c$; hence, by (3.2), $x \leq_P c$. This holds for all $x \in X$, thus, by (3.1), we obtain the inequality

$$a \leq_P c. \quad (3.5)$$

Hence, by (3.5) and (3.4), $a \leq b$. Therefore, $a = \sup X$ in R .

The proof for \bigwedge and \inf is similar.

Finally, the proof that $\langle R, f', g' \rangle$ is a pushout of \mathcal{D} is straightforward. \square

Notation 3.5. In the context of Proposition 3.4, in the case of a standard V-formation $\langle K, P, Q \rangle$, we shall write $R = P \sqcup_K Q$.

Part 2. D -valued posets and partial lattices

4. D -valued posets

We shall fix in this section a distributive lattice D with unit (largest element) 1.

The following definition is similar to the classical definition of a Boolean-valued model; see, for example, [16].

Definition 4.1. A D -valued poset is a nonempty set P , together with a map $P \times P \rightarrow D$, $\langle a, b \rangle \mapsto \|a \leq b\|$, that satisfies the following properties:

- (i) $\|a \leq a\| = 1$, for all $a \in P$.
- (ii) $\|a \leq b\| \wedge \|b \leq c\| \leq \|a \leq c\|$, for all $a, b, c \in P$.

If P is a D -valued poset, then we define $\|a = b\| = \|a \leq b\| \wedge \|b \leq a\|$, for all $a, b \in P$. Furthermore, for $a \in P$ and nonempty, finite subsets X and Y of P , we put

$$\|a \in Y\| = \bigvee_{y \in Y} \|a = y\|, \quad \|X \subseteq Y\| = \bigwedge_{x \in X} \|x \in Y\|,$$

and we put $\|X = Y\| = \|X \subseteq Y\| \wedge \|Y \subseteq X\|$.

We observe that a D -valued poset is *not* given with a partial ordering on P —there is no such thing as “the binary relation \leq on P .” Instead, $\|a \leq b\|$ denotes an *element of D* , as opposed to a *statement*.

Example 4.2. Let $\langle P, \leq \rangle$ be a poset. Then P can be canonically endowed with a structure of **2**-valued poset, by putting $\|a \leq b\| = 1$ if $a \leq b$, 0 otherwise.

Hence, the rule $\|a \leq b\| \wedge \|b \leq c\| \leq \|a \leq c\|$ may be interpreted as the D -valued version of the transitivity of the partial ordering.

We record below some basic facts about Boolean values. For the remainder of this section, we fix a D -valued poset P . In fact, many of the results below hold for D -valued models of equality (a set P with a map $\langle x, y \rangle \mapsto \|x = y\|$), with the same proofs.

Lemma 4.3. *The following assertions hold:*

- (i) $\|x = y\| \wedge \|y \in Z\| \leq \|x \in Z\|$, for all $x, y \in P$ and all $Z \in [P]_*^{<\omega}$.
- (ii) $\|x \in Y\| \wedge \|Y \subseteq Z\| \leq \|x \in Z\|$, for all $x \in P$ and all $Y, Z \in [P]_*^{<\omega}$.
- (iii) $\|X \subseteq Y\| \wedge \|Y \subseteq Z\| \leq \|X \subseteq Z\|$, for all $X, Y, Z \in [P]_*^{<\omega}$.
- (iv) $\|X = Y\| \wedge \|Y = Z\| \leq \|X = Z\|$, for all $X, Y, Z \in [P]_*^{<\omega}$.

Proof. (i) We compute:

$$\|x = y\| \wedge \|y \in Z\| = \bigvee_{z \in Z} \|x = y\| \wedge \|y = z\| \leq \bigvee_{z \in Z} \|x = z\| = \|x \in Z\|.$$

(ii) We compute, by using (i):

$$\|x \in Y\| \wedge \|Y \subseteq Z\| \leq \bigvee_{y \in Y} \|x = y\| \wedge \|Y \subseteq Z\| \leq \bigvee_{y \in Y} \|x = y\| \wedge \|y \in Z\| \leq \|x \in Z\|.$$

(iii) We compute, by using (ii):

$$\|X \subseteq Y\| \wedge \|Y \subseteq Z\| = \bigwedge_{x \in X} \|x \in Y\| \wedge \|Y \subseteq Z\| \leq \bigwedge_{x \in X} \|x \in Z\| = \|X \subseteq Z\|.$$

(iv) is an obvious consequence of (iii). \square

Lemma 4.4. *Let $a \in P$, let $X, Y \in [P]_*^{<\omega}$, let $\varphi(z, a)$ be one of the formulas $z \leq a$ or $a \leq z$. Then the following inequalities hold:*

- (i) $\|X \subseteq Y\| \wedge \bigwedge_{y \in Y} \|\varphi(y, a)\| \leq \|X \subseteq Y\| \wedge \bigwedge_{x \in X} \|\varphi(x, a)\|$.
- (ii) $\|X = Y\| \wedge \bigwedge_{y \in Y} \|\varphi(y, a)\| = \|X = Y\| \wedge \bigwedge_{x \in X} \|\varphi(x, a)\|$.

Proof. (i) Put $\gamma = \|X \subseteq Y\| \wedge \bigwedge_{y \in Y} \|\varphi(y, a)\|$. For $x \in X$,

$$\gamma \leq \|X \subseteq Y\| \leq \|x \in Y\| = \bigvee_{y \in Y} \|x = y\|,$$

so, to prove (i), it is sufficient to prove that $\gamma \wedge \|x = y\| \leq \|\varphi(x, a)\|$, for all $y \in Y$. But this follows from the fact that $\gamma \wedge \|x = y\| \leq \|x = y\| \wedge \|\varphi(y, a)\|$ and the definition of a D -valued poset.

(ii) follows immediately from (i). \square

Lemma 4.5. *Let $X, Y \in [P]_*^{<\omega}$. Then the following equality holds:*

$$\|X \subseteq Y\| = \bigvee_{\emptyset \subset Z \subseteq Y} \|X = Z\|.$$

Proof. For $\emptyset \subset Z \subseteq Y$, the inequality $\|X = Z\| \leq \|X \subseteq Y\|$ is clear. Conversely, we compute:

$$\|X \subseteq Y\| = \bigwedge_{x \in X} \bigvee_{y \in Y} \|x = y\| = \bigvee_{v: X \rightarrow Y} \bigwedge_{x \in X} \|x = v(x)\|,$$

so, to conclude the proof, it suffices to prove that for every map $v: X \rightarrow Y$, there exists Z such that $\emptyset \subset Z \subseteq Y$ and

$$\bigwedge_{x \in X} \|x = v(x)\| \leq \|X = Z\|. \quad (4.1)$$

We define Z as the range of v . So,

$$\bigwedge_{x \in X} \|x = v(x)\| \leq \bigwedge_{x \in X} \|x \in Z\| = \|X \subseteq Z\|.$$

Furthermore, if $z \in Z$, so, $z = v(x^*)$ for some $x^* \in X$, then

$$\bigwedge_{x \in X} \|x = v(x)\| \leq \|x^* = v(x^*)\| \leq \|z \in X\|;$$

thus $\bigwedge_{x \in X} \|x = v(x)\| \leq \|Z \subseteq X\|$; so, finally, (4.1) holds. This concludes the proof. \square

Every D -valued poset P can be “localized” at every prime filter of D , in a classical fashion that we shall recall here. Let G be any *filter* of D , that is, a nonempty upper subset of D closed under finite meet. We define binary relations, \leq_G and \equiv_G , on P , by the rule

$$\begin{aligned} a \leq_G b &\iff \|a \leq b\| \in G, \\ a \equiv_G b &\iff \|a = b\| \in G, \end{aligned}$$

for all $a, b \in P$. It is easy to verify that the relation \leq_G is a preordering on P , and that \equiv_G is the associated equivalence relation. Hence, the quotient structure $P/G = \langle P, \leq_G \rangle / \equiv_G$ may be endowed with a partial ordering, defined by the rule

$$a/G \leq b/G \iff a \leq_G b,$$

for all $a, b \in P$, where we write, of course, $a/G = a/\equiv_G$.

The abundance of prime filters may be recorded in the following classical result, that we shall use most of the time without mentioning:

Lemma 4.6. *Let $a, b \in D$. Then $a \leq b$ iff $a \in G$ implies that $b \in G$ for all prime filters G of D .*

As a rule, handling D -valued posets is very similar to handling Boolean-valued posets. We point out two important differences with the classical context:

- The “value set” D is no longer a complete Boolean algebra as it is usually the case in the theory of Boolean-valued models. It is only a distributive lattice, not even necessarily complete.
- No analogue of “fullness,” as it is ordinarily defined for Boolean models, will be assumed or even considered throughout this paper.

5. D -valued partial lattices

Definition 5.1. A D -valued partial lattice is a D -valued poset P , endowed with two maps from $P \times [P]_*^{<\omega} \rightarrow D$, denoted respectively by $\langle a, X \rangle \mapsto \|a = \bigvee X\|$ and $\langle a, X \rangle \mapsto \|a = \bigwedge X\|$, such that for all $a, b \in P$ and all $X, Y \in [P]_*^{<\omega}$, the following equalities hold:

- (1) $\|a = \bigvee X\| \wedge \|a \leq b\| = \|a = \bigvee X\| \wedge \bigwedge_{x \in X} \|x \leq b\|$;
- (1*) $\|a = \bigwedge X\| \wedge \|b \leq a\| = \|a = \bigwedge X\| \wedge \bigwedge_{x \in X} \|b \leq x\|$;
- (2) $\|a = \bigvee X\| \wedge \|X = Y\| \leq \|a = \bigvee Y\|$;
- (2*) $\|a = \bigwedge X\| \wedge \|X = Y\| \leq \|a = \bigwedge Y\|$;
- (3) $\|a = \bigvee X\| \wedge \|a = b\| \leq \|b = \bigvee X\|$;
- (3*) $\|a = \bigwedge X\| \wedge \|a = b\| \leq \|b = \bigwedge X\|$.

Example 5.2. Every partial lattice P can be viewed as a $\mathbf{2}$ -valued poset, as in Example 4.2. This structure can be extended to a structure of $\mathbf{2}$ -valued partial lattice, by putting

$$\|a = \bigvee X\| = \begin{cases} 1 & \text{if } a = \bigvee X, \\ 0 & \text{otherwise,} \end{cases}$$

and similarly for \bigwedge .

For the remainder of this section, we shall fix a D -valued partial lattice P .

Lemma 5.3. *Let $a, b \in P$ and let $X \in [P]_*^{<\omega}$. Then the following assertions hold:*

- (i) $\|a = \bigvee X\| \wedge \|b = \bigvee X\| \leq \|a = b\|$;
- (ii) $\|a = \bigwedge X\| \wedge \|b = \bigwedge X\| \leq \|a = b\|$.

Proof. We only prove (i). Put $\gamma = \|a = \bigvee X\| \wedge \|b = \bigvee X\|$. By (1) of Definition 5.1,

$$\|b = \bigvee X\| \wedge \bigwedge_{x \in X} \|x \leq b\| = \|b = \bigvee X\| \wedge \|b \leq b\| = \|b = \bigvee X\|,$$

thus $\gamma \leq \|b = \bigvee X\| \leq \bigwedge_{x \in X} \|x \leq b\|$. Furthermore,

$$\begin{aligned} \gamma \wedge \|a \leq b\| &= \gamma \wedge \bigwedge_{x \in X} \|x \leq b\| \quad (\text{by (1) of Definition 5.1}) \\ &= \gamma \quad (\text{by the above paragraph}), \end{aligned}$$

so $\gamma \leq \|a \leq b\|$. Symmetrically, $\gamma \leq \|b \leq a\|$, so the conclusion follows. \square

If G is a filter of D , we have seen that we can define a quotient poset P/G . We shall now show how to extend the structure of P/G to a structure of partial lattice.

Definition 5.4. Let $X \in [P/G]_*^{<\omega}$ and let $a \in P/G$. We define $a = \bigvee X$ (respectively $a = \bigwedge X$) to hold, if there are $a \in P$ and $X \in [P]_*^{<\omega}$ such that $a = a/G$, $X = X/G$, and $\|a = \bigvee X\| \in G$ (respectively $\|a = \bigwedge X\| \in G$).

As an immediate consequence of Definition 5.1(2)–(3*), we obtain the following lemma.

Lemma 5.5. Let $a \in P$, let $X \in [P]_*^{<\omega}$. Then $a/G = \bigvee X/G$ (respectively $a/G = \bigwedge X/G$) iff $\|a = \bigvee X\| \in G$ (respectively $\|a = \bigwedge X\| \in G$).

Proposition 5.6. The poset P/G , endowed with \bigvee and \bigwedge of Definition 5.4, is a partial lattice.

Proof. We first have to prove that \bigvee and \bigwedge are functions. We do it for \bigvee . So let $X \in [P/G]_*^{<\omega}$ and let $a, b \in P/G$ such that $a = \bigvee X$ and $b = \bigvee X$. Let $a, b \in P$ and let $X \in [P]_*^{<\omega}$ such that $a = a/G$, $b = b/G$, and $X = X/G$. By Lemma 5.5, both $\|a = \bigvee X\|$ and $\|b = \bigvee X\|$ belong to G , hence, by Lemma 5.3, $\|a = b\| \in G$, so $a = b$. Hence \bigvee is a function on P/G . The same argument applies to \bigwedge .

To conclude the proof, it is sufficient to prove that for $a \in P/G$ and $X \in [P/G]_*^{<\omega}$, $a = \bigvee X$ implies that $a = \sup X$ (for the partial ordering of P/G), and similarly for \bigwedge . We present the proof for \bigvee . Let $a \in P$ and $X \in [P]_*^{<\omega}$ such that $a = a/G$ and $X = X/G$. By Lemma 5.5, $\|a = \bigvee X\| \in G$. For $x \in X$, it follows from Definition 5.1(1) that

$$\|a = \bigvee X\| = \|a = \bigvee X\| \wedge \|a \leq a\| \leq \|a = \bigvee X\| \wedge \|x \leq a\| \leq \|x \leq a\|,$$

so $\|x \leq a\| \in G$, that is, $x/G \leq a/G = a$. So, $X \leq a$. Now let $b \in P/G$ such that $X \leq b$. Pick $b \in P$. For $x \in X$, $x/G \leq b/G = b$, so $\|x \leq b\| \in G$; hence $\bigwedge_{x \in X} \|x \leq b\| \in G$. By Definition 5.1(1),

$$\|a = \bigvee X\| \wedge \|a \leq b\| = \|a = \bigvee X\| \wedge \bigwedge_{x \in X} \|x \leq b\| \in G,$$

hence $\|a \leq b\| \in G$, that is, $a \leq b$. So we have proved that $a = \sup X$. \square

6. Join-samples and meet-samples

Let D be a distributive lattice with unit, let P be a D -valued partial lattice. We introduce one of the most important definitions of the whole paper:

Definition 6.1. Let X be a nonempty finite subset of P . A *join-sample* (respectively *meet-sample*) of X is a nonempty finite subset U of P such that

$$\begin{aligned} \|x = \bigvee X\| &\leq \bigvee_{u \in U} \|u = \bigvee X\|, \quad \text{for all } x \in P \\ \text{(respectively } \|x = \bigwedge X\| &\leq \bigvee_{u \in U} \|u = \bigwedge X\|, \quad \text{for all } x \in P). \end{aligned}$$

Definition 6.2. A D -valued partial lattice P is *finitely join-sampled* (respectively *finitely meet-sampled*), if every nonempty finite subset of P has a join-sample (respectively a meet-sample). We say that P is *finitely sampled*, if it is both finitely join-sampled and finitely meet-sampled.

Of course, if U is a join-sample of X and V is a meet-sample of X , then $U \cup V$ (or anything larger) is both a join-sample and a meet-sample of X .

Lemma 6.3. Let $X, U, V \in [P]_*^{<\omega}$.

(i) If U and V are join-samples of X , then the equality

$$\bigvee_{u \in U} \|a \leq u\| \wedge \|u = \bigvee X\| = \bigvee_{v \in V} \|a \leq v\| \wedge \|v = \bigvee X\|$$

holds, for all $a \in P$.

(ii) If U and V are meet-samples of X , then the equality

$$\bigvee_{u \in U} \|u \leq a\| \wedge \|u = \bigwedge X\| = \bigvee_{v \in V} \|v \leq a\| \wedge \|v = \bigwedge X\|$$

holds, for all $a \in P$.

Proof. We provide a proof for (i); (ii) is dual. For $u \in U$,

$$\begin{aligned} &\|a \leq u\| \wedge \|u = \bigvee X\| \\ &= \bigvee_{v \in V} (\|a \leq u\| \wedge \|u = \bigvee X\| \wedge \|v = \bigvee X\|) \quad (\text{because } V \text{ is a join-sample of } X) \\ &\leq \bigvee_{v \in V} (\|a \leq u\| \wedge \|u = v\| \wedge \|v = \bigvee X\|) \quad (\text{by Lemma 5.3}) \end{aligned}$$

$$\leq \bigvee_{v \in V} \|a \leq v\| \wedge \|v = \bigvee X\|;$$

hence $\bigvee_{u \in U} \|a \leq u\| \wedge \|u = \bigvee X\| \leq \bigvee_{v \in V} \|a \leq v\| \wedge \|v = \bigvee X\|$. The proof of the converse inequality is similar. \square

Lemma 6.3 makes it possible to define, for all $a \in P$ and all $X \in [P]_*^{<\omega}$,

$$\begin{aligned} \|a \leq \bigvee X\| &= \bigvee_{u \in U} \|a \leq u\| \wedge \|u = \bigvee X\|, \quad \text{for every join-sample } U \text{ of } X, \\ \|\bigwedge X \leq a\| &= \bigvee_{u \in U} \|u \leq a\| \wedge \|u = \bigwedge X\|, \quad \text{for every meet-sample } U \text{ of } X. \end{aligned}$$

We recall that a filter G of D is *prime*, if $x \vee y \in G$ implies that $x \in G$ or $y \in G$, for all $x, y \in D$.

Lemma 6.4 (The basic truth lemma). *Assume that P is finitely sampled. Let $\varphi(z, Z)$ be one of the following formulas:*

$$z = \bigvee Z; \quad z \leq \bigvee Z; \quad z = \bigwedge Z; \quad \bigwedge Z \leq z.$$

Let $a \in P$, let $X \in [P]_^{<\omega}$, let G be a prime filter of D . Then the following equivalence holds:*

$$P/G \text{ satisfies } \varphi(a/G, X/G) \quad \text{iff} \quad \|\varphi(a, X)\| \in G.$$

Proof. By duality, it is sufficient to prove the result in case $\varphi(z, Z)$ is either $z = \bigvee Z$ or $z \leq \bigvee Z$. The first case follows from Lemma 5.5. So, suppose that $\varphi(z, Z)$ is $z \leq \bigvee Z$.

Let U be a join-sample of X . Suppose first that $a/G \leq \bigvee X/G$ (in P/G). In particular, $\bigvee (X/G)$ is defined, so, by Definition 5.4 and by Lemma 5.5, there exists $a' \in P$ such that

$$\|a' = \bigvee X\| \in G, \tag{6.1}$$

$$\|a \leq a'\| \in G. \tag{6.2}$$

Since U is a join-sample of X , $\|a' = \bigvee X\| \leq \bigvee_{u \in U} \|u = \bigvee X\|$, thus, since G is prime and U is finite, there exists, by (6.1), $u \in U$ such that

$$\|u = \bigvee X\| \in G. \tag{6.3}$$

From (6.1) and Lemma 5.3, it follows that $\|a' = u\| \in G$, thus, by (6.2), $\|a \leq u\| \in G$. Hence, by (6.3), $\|a \leq \bigvee X\| \in G$.

Conversely, suppose that $\|a \leq \bigvee X\| \in G$. Since U is finite and G is prime, there exists $u \in U$ such that $\|a \leq u\| \in G$ and $\|u = \bigvee X\| \in G$. Hence, by Lemma 5.5, $a/G \leq u/G$ and $u/G = \bigvee X/G$, so $a/G \leq \bigvee X/G$. \square

Further analogues of Lemma 6.4 will be met in Lemmas 7.5, 7.8, Propositions 9.4, 11.8, Corollary 11.7, and Lemma 14.1.

7. Ideal and filter samples

In this section, we fix a distributive lattice D with unit and a D -valued partial lattice P .

Definition 7.1. Let $X \in [P]_*^{<\omega}$. An $(\text{Id}\cap)$ -sample of X is an element U of $[P]_*^{<\omega}$ such that

$$\bigwedge_{x \in X} \|a \leq x\| = \bigvee_{u \in U} \left(\|a \leq u\| \wedge \bigwedge_{x \in X} \|u \leq x\| \right) \quad (7.1)$$

holds for all $a \in P$.

Dually, a $(\text{Fil}\cap)$ -sample of X is an element U of $[P]_*^{<\omega}$ such that

$$\bigwedge_{x \in X} \|x \leq a\| = \bigvee_{u \in U} \left(\|u \leq a\| \wedge \bigwedge_{x \in X} \|x \leq u\| \right) \quad (7.2)$$

holds for all $a \in P$.

We observe that the \geq half of both equalities (7.1) and (7.2) always holds, thus it is sufficient to verify the \leq half. As a consequence of this, we observe that *every finite subset of P that contains an $(\text{Id}\cap)$ -sample of X is an $(\text{Id}\cap)$ -sample of X .*

Definition 7.2. We say that P has $(\text{Id}\cap)$ (respectively $(\text{Fil}\cap)$), if every pair of elements of P has an $(\text{Id}\cap)$ -sample (respectively a $(\text{Fil}\cap)$ -sample).

We state without proof the following easy result, that will not be used later.

Proposition 7.3. If P has $(\text{Id}\cap)$ (respectively $(\text{Fil}\cap)$), then every nonempty finite subset of P has an $(\text{Id}\cap)$ -sample (respectively a $(\text{Fil}\cap)$ -sample).

The last two finiteness properties about P that we shall consider are harder to define. To prepare for this task, we first define new D -valued functions on P .

Definition 7.4. Suppose that P is finitely join-sampled. For $a \in P$, for nonempty, finite subsets X and U of P , and for $n < \omega$, we define an element $\|a \in \text{Id}_n(X, U)\|$ of D , by induction on n , as follows:

- (i) $\|a \in \text{Id}_0(X, U)\| = \|a \in \downarrow X\| = \bigvee_{x \in X} \|a \leq x\|$.
- (ii) The induction step:

$$\|a \in \text{Id}_{n+1}(X, U)\| = \|a \in \text{Id}_n(X, U)\| \vee \bigvee_{\emptyset \subset Z \subseteq U} \left(\|a \leq \bigvee Z\| \wedge \|Z \subseteq \text{Id}_n(X, U)\| \right),$$

where we put $\|Z \subseteq \text{Id}_n(X, U)\| = \bigwedge_{z \in Z} \|z \in \text{Id}_n(X, U)\|$.

Dually, suppose that P is finitely meet-sampled. For $a \in P$, for nonempty, finite subsets X and U of P , and for $n < \omega$, we define an element $\|a \in \text{Id}_n(X, U)\|$ of D , by induction on n , as follows:

- (i*) $\|a \in \text{Fil}_0(X, U)\| = \|a \in \uparrow X\| = \bigvee_{x \in X} \|x \leq a\|$.
- (ii*) The induction step:

$$\|a \in \text{Fil}_{n+1}(X, U)\| = \|a \in \text{Fil}_n(X, U)\| \vee \bigvee_{\emptyset \subset Z \subseteq U} \left\| \bigwedge Z \leq a \right\| \wedge \|Z \subseteq \text{Fil}_n(X, U)\|,$$

where we put $\|Z \subseteq \text{Fil}_n(X, U)\| = \bigwedge_{z \in Z} \|z \in \text{Fil}_n(X, U)\|$.

We observe that the condition that P be finitely join- or meet-sampled is necessary in order to *define* the elements $\|a \in \text{Id}_n(X, U)\|$ and $\|a \in \text{Fil}_n(X, U)\|$, since the elements $\|a \leq \bigvee Z\|$ and $\|\bigwedge Z \leq a\|$ need to be defined. Our next result relates the D -valued Id_n and Fil_n with their corresponding classical versions, see Definition 2.9.

Lemma 7.5 (Truth lemma for $\text{Id}_n(X, U)$ and $\text{Fil}_n(X, U)$). *Let $a \in P$, let $X, U \in [P]_*^{<\omega}$, and let G be a prime filter of D . Then the following assertions hold:*

- (i) *Suppose that P is finitely join-sampled. Then $a/G \in \text{Id}_n(X/G, U/G)$ in P/G iff $\|a \in \text{Id}_n(X, U)\| \in G$, for any $n < \omega$.*
- (ii) *Suppose that P is finitely meet-sampled. Then $a/G \in \text{Fil}_n(X/G, U/G)$ in P/G iff $\|a \in \text{Fil}_n(X, U)\| \in G$, for any $n < \omega$.*

Proof. We provide a proof for (i). We argue by induction on n . The result for $n = 0$ follows immediately from the finiteness of X and the fact that G is prime.

Now suppose that the statement proved for n ; we will prove it for $n + 1$. Suppose first that $\|a \in \text{Id}_{n+1}(X, U)\| \in G$. Since $\mathcal{P}(U)$ is finite and since G is prime, we have that either $\|a \in \text{Id}_n(X, U)\| \in G$, or there exists a nonempty finite subset Z of U such that $\|a \leq \bigvee Z\| \in G$ and $\|Z \subseteq \text{Id}_n(X, U)\| \in G$. In the first case, it follows from the induction hypothesis that $a/G \in \text{Id}_n(X/G, U/G) \subseteq \text{Id}_{n+1}(X/G, U/G)$, so we are done. In the second case, $a/G \leq \bigvee (Z/G)$ by Lemma 6.4, $Z/G \subseteq \text{Id}_n(X/G, U/G)$ by the induction hypothesis, and $\emptyset \subset Z/G \subseteq U/G$, hence, $a/G \in \text{Id}_{n+1}(X/G, U/G)$.

Conversely, suppose that $a/G \in \text{Id}_{n+1}(X/G, U/G)$. If $a/G \in \text{Id}_n(X/G, U/G)$, then, by the induction hypothesis, $\|a \in \text{Id}_n(X, U)\| \in G$, hence $\|a \in \text{Id}_{n+1}(X, U)\| \in G$. Otherwise, there exists a nonempty $Z \subseteq U/G$ such that $a/G \leq \bigvee Z$ and $Z \subseteq \text{Id}_n(X/G, U/G)$. Since $\emptyset \subset Z \subseteq U/G$, there exists a nonempty subset Z of U such that $Z = Z/G$. So $a/G \leq \bigvee Z/G$, thus, by Lemma 6.4, $\|a \leq \bigvee Z\| \in G$. Since $Z/G = Z \subseteq \text{Id}_n(X/G, U/G)$, it follows from the induction hypothesis that $\|Z \subseteq \text{Id}_n(X, U)\| \in G$. Since $\emptyset \subset Z \subseteq U$, Z witnesses the fact that $\|a \in \text{Id}_{n+1}(X, U)\| \in G$. \square

Definition 7.6. Let X be a nonempty finite subset of P . An $(\text{Id}\vee)$ -sample (respectively $(\text{Fil}\vee)$ -sample) of X is a nonempty finite subset U of P such that there exists $n < \omega$ such that

$$\|a \in \text{Id}_n(X, U)\| = \|a \in \text{Id}_{n+1}(X, Y)\|$$

$$\text{(respectively } \|a \in \text{Fil}_n(X, U)\| = \|a \in \text{Fil}_{n+1}(X, Y)\|),$$

for all $a \in P$ and all $Y \in [P]_*^{<\omega}$ containing U .

We call any such n an *ideal index* (respectively *filter index*) of $\langle X, U \rangle$.

If U is an $(\text{Id}\vee)$ -sample of X , with ideal index n , then it is easy to verify that $\|a \in \text{Id}_k(X, Y)\| = \|a \in \text{Id}_n(X, U)\|$, for all $a \in P$, all $k \geq n$, and all finite $Y \supseteq U$. Hence this expression is independent of the chosen sample U and index n , we denote it by $\|a \in \text{Id}(X)\|$.

Dually, we define $\|a \in \text{Fil}(X)\|$ as the common value of $a \in \text{Fil}_n(X, U)$, for every $(\text{Fil}\vee)$ -sample U of X , with filter index n .

Definition 7.7. We say that P has $(\text{Id}\vee)$ (respectively $(\text{Fil}\vee)$), if P is finitely join-sampled (respectively finitely meet-sampled) and every nonempty finite subset of P has an $(\text{Id}\vee)$ -sample (respectively a $(\text{Fil}\vee)$ -sample).

As an easy consequence of the remarks following Definition 2.9 and of Lemma 7.5, we obtain the following.

Lemma 7.8 (Truth lemma for $\text{Id}(X)$ and $\text{Fil}(X)$). *Let $a \in P$, let $X \in [P]_*^{<\omega}$, and let G be a prime filter of D . Then the following equivalences hold:*

(i) *Suppose that P has $(\text{Id}\vee)$. Then*

$$a/G \in \text{Id}(X/G) \text{ in } P/G \quad \text{iff} \quad \|a \in \text{Id}(X)\| \in G.$$

(ii) *Suppose that P has $(\text{Fil}\vee)$. Then*

$$a/G \in \text{Fil}(X/G) \text{ in } P/G \quad \text{iff} \quad \|a \in \text{Fil}(X)\| \in G.$$

The definition of $(\text{Id}\vee)$ and of $(\text{Fil}\vee)$ for D -valued partial lattice presented in Definition 7.7 is quite unwieldy, because it involves the Boolean values $\|a \in \text{Id}_n(X, U)\|$ or $\|a \in \text{Fil}_n(X, U)\|$ presented in Definition 7.6. However, Lemma 7.5 makes it possible to find a useful equivalent form.

Lemma 7.9. *The finitely join-sampled D -valued partial lattice P has $(\text{Id}\vee)$ iff for all $X \in [P]_*^{<\omega}$, there are $U \in [P]_*^{<\omega}$ and $n < \omega$ such that*

$$\text{Id}_n(X/G, U/G) = \text{Id}_{n+1}(X/G, Y/G),$$

for every $Y \in [P]_^{<\omega}$ containing U and every prime filter G of D . The dual statement holds, about $(\text{Fil}\vee)$ and Fil_n , for finitely meet-sampled P .*

8. Affine lower and upper functions; ideal and filter functions

Let D be a distributive lattice with unit. The D -valued analogue of the notions of a lower set and an upper set are provided by the following definition.

Definition 8.1. Let P be a D -valued poset. A map $f : P \rightarrow D$ is a *lower function*, if $f(y) \wedge \|x \leq y\| \leq f(x)$, for all $x, y \in P$. Dually, f is an *upper function*, if $f(x) \wedge \|x \leq y\| \leq f(y)$, for all $x, y \in P$.

For example, if P is a poset, viewed, as in Example 4.2, with its canonical structure of $\mathbf{2}$ -valued poset, then the lower (respectively upper) functions on P are exactly the characteristic functions of the lower (respectively upper) subsets of P .

It is obvious that for $a \in P$, the map $x \mapsto \|x \leq a\|$ (respectively $x \mapsto \|a \leq x\|$) is a lower function (respectively upper function) on P —we shall call these functions *principal lower functions* (respectively *principal upper functions*). Furthermore, any constant function is both a lower function and an upper function, and any finite meet or join of lower functions (respectively upper functions) is a lower function (respectively an upper function). This gives a class of “simple” lower functions and upper functions, an analogue of finitely generated lower subsets or upper subsets of a poset.

Definition 8.2. Let P be a D -valued poset. An *affine lower function* on P is a map $f : P \rightarrow D$ defined by a rule of the form

$$f(x) = \bigvee_{i < n} \|x \leq u_i\| \wedge \alpha_i, \quad \text{for all } x \in P,$$

where $n \in \omega \setminus \{0\}$, $u_0, \dots, u_{n-1} \in P$, and $\alpha_0, \dots, \alpha_{n-1} \in D$. Dually, an *affine upper function* on P is a map $f : P \rightarrow D$ defined by a rule of the form

$$f(x) = \bigvee_{i < n} \|u_i \leq x\| \wedge \alpha_i, \quad \text{for all } x \in P,$$

where $n \in \omega \setminus \{0\}$, $u_0, \dots, u_{n-1} \in P$, and $\alpha_0, \dots, \alpha_{n-1} \in D$.

In particular, any affine lower function is a lower function, and any affine upper function is an upper function.

Definition 8.3. Let P be a D -valued partial lattice. An *ideal function* on P is a lower function $f : P \rightarrow D$ such that

$$\|a = \bigvee X\| \wedge \bigwedge_{x \in X} f(x) \leq f(a), \quad \text{for all } a \in P \text{ and } X \in [P]_*^{<\omega}.$$

Dually, a *filter function* on P is an upper function $f : P \rightarrow D$ such that

$$\|a = \bigwedge X\| \wedge \bigwedge_{x \in X} f(x) \leq f(a), \quad \text{for all } a \in P \text{ and } X \in [P]_*^{<\omega}.$$

An *affine ideal function* on P is a function that is simultaneously an affine lower function and an ideal function on P . Dually, an *affine filter function* on P is a function that is simultaneously an affine upper function and a filter function on P .

We observe that the set of all ideal functions (respectively filter functions) on P is closed under component-wise meet, but not under component-wise join as a rule, just the same way as the *union* of two ideals of a partial lattice is not necessarily an ideal.

Example 8.4. Every partial lattice P can be viewed as a $\mathbf{2}$ -valued partial lattice, see Example 5.2. If I is an ideal of P , then the characteristic function of I is an ideal function on P , and, dually, a similar statement holds for filters.

Lemma 8.5. Let P be a D -valued partial lattice, let $f : P \rightarrow D$.

(i) If P is finitely join-sampled and f is an ideal function, then

$$\left\| a \leq \bigvee X \right\| \wedge \bigwedge_{x \in X} f(x) \leq f(a), \quad \text{for all } a \in P \text{ and } X \in [P]_*^{<\omega}.$$

(ii) If P is finitely meet-sampled and f is a filter function, then

$$\left\| \bigwedge X \leq a \right\| \wedge \bigwedge_{x \in X} f(x) \leq f(a), \quad \text{for all } a \in P \text{ and } X \in [P]_*^{<\omega}.$$

Proof. We provide a proof for (i). Let U be a join-sample of X . Then, for $a \in P$,

$$\begin{aligned} \left\| a \leq \bigvee X \right\| \wedge \bigwedge_{x \in X} f(x) &= \bigvee_{u \in U} \left(\left\| a \leq u \right\| \wedge \left\| u = \bigvee X \right\| \wedge \bigwedge_{x \in X} f(x) \right) \\ &\leq \bigvee_{u \in U} \left\| a \leq u \right\| \wedge f(u) \quad (\text{because } f \text{ is an ideal function}) \\ &\leq f(a) \quad (\text{because } f \text{ is a lower function}). \quad \square \end{aligned}$$

In a D -valued partial lattice P , it is easy to prove that any principal lower function is an affine ideal function, and any principal upper function is an affine filter function. Our next result provides extensions of this simple fact.

Proposition 8.6. Let P be a D -valued partial lattice, let X and U be nonempty finite subsets of P . Then the following assertions hold:

(i) Suppose that P is finitely join-sampled. Then the map $a \mapsto \|a \in \text{Id}_n(X, U)\|$ is an affine lower function on P . Furthermore, if P has $(\text{Id}\vee)$, then the map $a \mapsto \|a \in \text{Id}(X)\|$ is an affine ideal function on P .

- (ii) Suppose that P is finitely meet-sampled. Then the map $a \mapsto \|a \in \text{Fil}_n(X, U)\|$ is an affine upper function on P . Furthermore, if P has $(\text{Fil}\vee)$, then the map $a \mapsto \|a \in \text{Fil}(X)\|$ is an affine filter function on P .

Proof. We provide a proof for (i). For $n < \omega$, let $f_n : a \mapsto \|a \in \text{Id}_n(X, U)\|$. We prove, by induction on n , that f_n is an affine lower function.

For $n = 0$, $f_0(a) = \bigvee_{x \in X} \|a \leq x\|$ for all a , thus f_0 is an affine lower function.

Before proceeding to the induction step, we prove a claim.

Claim 1. The map $a \mapsto \|a \leq \bigvee Y\|$ is an affine lower function, for all $Y \in [P]_*^{\leq \omega}$.

Proof of Claim 1. Let V be a join-sample of Y . Then

$$\left\| a \leq \bigvee Y \right\| = \bigvee_{v \in V} \|a \leq v\| \wedge \beta_v, \quad \text{for all } a \in P \quad (\text{use Lemma 6.3(i)}),$$

where we put $\beta_v = \|v = \bigvee Y\|$, for all $v \in V$. \square

Now suppose that f_n is an affine lower function on P . Then

$$f_{n+1}(a) = f_n(a) \vee \bigvee_{\emptyset \subset Z \subseteq U} \left\| a \leq \bigvee Z \right\| \wedge \gamma_Z,$$

for all $a \in P$, where we put $\gamma_Z = \|Z \subseteq \text{Id}_n(X, U)\|$, for all nonempty $Z \subseteq U$. Therefore, by Claim 1 and the induction hypothesis, f_{n+1} is an affine lower function. So all f_n are affine lower functions.

Now let $f : a \mapsto \|a \in \text{Id}(X)\|$. Suppose that P has $(\text{Id}\vee)$. Let U be an $(\text{Id}\vee)$ -sample of X , with index n . So $\|a \in \text{Id}(X)\| = \|a \in \text{Id}_n(X, U)\|$, for all $a \in P$. Hence f is an affine lower function. So, to conclude the proof, it is sufficient to prove that f is an ideal function. So let $Z \in [P]_*^{\leq \omega}$, let $a \in P$. We compute:

$$\begin{aligned} \left\| a = \bigvee Z \right\| &\wedge \bigwedge_{z \in Z} f(z) \\ &= \left\| a = \bigvee Z \right\| \wedge \bigwedge_{z \in Z} \|z \in \text{Id}_n(X, U)\| \\ &= \left\| a = \bigvee Z \right\| \wedge \|Z \subseteq \text{Id}_n(X, U)\| \\ &\leq \left\| a = \bigvee Z \right\| \wedge \|Z \subseteq \text{Id}_n(X, U \cup Z)\| \\ &\leq \|a \in \text{Id}_{n+1}(X, U \cup Z)\| \quad \left(\text{because } \left\| a = \bigvee Z \right\| \leq \left\| a \leq \bigvee Z \right\| \right) \\ &= f(a). \end{aligned}$$

So f is an ideal function on P . \square

Notation 8.7. Let P be a D -valued partial lattice. We denote by $\mathcal{I}_{\text{aff}}(P)$ (respectively $\mathcal{F}_{\text{aff}}(P)$) the set of all affine ideal functions (respectively affine filter functions) on P , partially ordered component-wise.

For the remainder of this section, we assume that P is a D -valued partial lattice.

Notation 8.8. Let $f : P \rightarrow D$. If there exists a least ideal (respectively filter) function g such that $f \leq g$, then we denote this function by f^{Id} (respectively f^{Fil}).

We observe that $f \leq f^{\text{Id}}$ and also $f \leq f^{\text{Fil}}$.

Lemma 8.9. Let $X \in [P]_*^{<\omega}$.

- (i) Let P have $(\text{Id}\vee)$ and let $f : a \mapsto \|a \in \downarrow X\| \wedge \alpha$. Then $f^{\text{Id}} : a \mapsto \|a \in \text{Id}(X)\| \wedge \alpha$.
- (ii) Let P have $(\text{Fil}\vee)$ and let $f : a \mapsto \|a \in \uparrow X\| \wedge \alpha$. Then $f^{\text{Fil}} : a \mapsto \|a \in \text{Fil}(X)\| \wedge \alpha$.

Proof. We provide a proof for (i). Let $g : a \mapsto \|a \in \text{Id}(X)\| \wedge \alpha$. It is obvious that $f \leq g$. By Proposition 8.6, g is an affine ideal function on P .

It remains to prove that $g \leq h$, for every ideal function h on P such that $f \leq h$. Since g has the form $a \mapsto \|a \in \text{Id}_n(X, U)\| \wedge \alpha$, for some U and some n , it suffices to prove that

$$\|a \in \text{Id}_n(X, U)\| \wedge \alpha \leq h(a), \quad \text{for all } a \in P, \ U \in [P]_*^{<\omega}, \text{ and } n < \omega. \quad (8.1)$$

For $n = 0$, $\|a \in \text{Id}_n(X, U)\| \wedge \alpha = f(a) \leq h(a)$, so (8.1) holds. Assume that (8.1) holds for n . For nonempty $Z \subseteq U$, we compute:

$$\|a \leq \bigvee Z\| \wedge \|Z \subseteq \text{Id}_n(X, U)\| \wedge \alpha \leq \|a \leq \bigvee Z\| \wedge \bigwedge_{z \in Z} h(z)$$

(by the induction hypothesis)

$$\leq h(a)$$

by Lemma 8.5. Hence $\|a \in \text{Id}_{n+1}(X, U)\| \wedge \alpha \leq h(a)$, for all $a \in P$. This concludes the proof of (8.1). \square

As a consequence, f^{Id} can be computed explicitly, for any affine lower function f (and dually).

Proposition 8.10. Let $n \in \omega \setminus \{0\}$, $u_0, \dots, u_{n-1} \in P$, and $\alpha_0, \dots, \alpha_{n-1} \in D$. For all non-empty $I \subseteq n$, we put

$$u^{(I)} = \{u_i \mid i \in I\}, \quad \alpha_{(I)} = \bigwedge_{i \in I} \alpha_i.$$

(i) Suppose that P has $(\text{Id}\vee)$. Let $f : a \mapsto \bigvee_{i < n} \|a \leq u_i\| \wedge \alpha_i$. Then f^{Id} is defined, and

$$f^{\text{Id}}(a) = \bigvee_{\emptyset \subset I \subseteq n} \|a \in \text{Id}(u^{(I)})\| \wedge \alpha_{(I)},$$

for all $a \in P$. In particular, f^{Id} is an affine ideal function.

(ii) Suppose that P has $(\text{Fil}\vee)$. Let $f : a \mapsto \bigvee_{i < n} \|u_i \leq a\| \wedge \alpha_i$. Then f^{Fil} is defined, and

$$f^{\text{Fil}}(a) = \bigvee_{\emptyset \subset I \subseteq n} \|a \in \text{Fil}(u^{(I)})\| \wedge \alpha_{(I)},$$

for all $a \in P$. In particular, f^{Fil} is an affine filter function.

Proof. We provide a proof for (i). By Lemma 8.9, for $\emptyset \subset I \subseteq n$, the map $g_I : a \mapsto \|a \in \text{Id}(u^{(I)})\| \wedge \alpha_{(I)}$ is an affine ideal function, so $g = \bigvee_{\emptyset \subset I \subseteq n} g_I$ is an affine lower function on P . Furthermore, $g_{\{i\}}(a) = \|a \leq u_i\| \wedge \alpha_i$, for all $i < n$ and all $a \in P$, thus $f \leq g$. Let h be an ideal function on P such that $f \leq h$. In order to verify that $g \leq h$, it suffices to verify that $g_I \leq h$ for all nonempty $I \subseteq n$. For $a \in P$,

$$\|a \in \downarrow u^{(I)}\| \wedge \alpha_{(I)} = \bigvee_{i \in I} \|a \leq u_i\| \wedge \alpha_{(I)} \leq f(a) \leq h(a).$$

Therefore, by Lemma 8.9, $g_I \leq h$. This holds for all I , therefore, $g \leq h$.

To conclude the proof, it suffices to prove that g is an ideal function on P . So, let $a \in P$ and let $X \in [P]_*^{\leq \omega}$. We shall prove that

$$\|a = \bigvee X\| \wedge \bigwedge_{x \in X} g(x) \leq g(a). \quad (8.2)$$

To prove (8.2), it suffices to prove that $\|a = \bigvee X\| \wedge \bigwedge_{x \in X} g(x) \in G$ implies that $g(a) \in G$, for any prime filter G of D . By Lemmas 6.4 and 7.8 and by the definition of g ,

$$a/G = \bigvee X/G, \quad (8.3)$$

and, for all $x \in X$, there exists a nonempty $I_x \subseteq n$ such that

$$\alpha_{(I_x)} \in G, \quad (8.4)$$

$$\|x \in \text{Id}(u^{(I_x)})\| \in G. \quad (8.5)$$

Now put $I = \bigcup_{x \in X} I_x$. Then, by (8.4), $\alpha_{(I)} = \bigwedge_{x \in X} \alpha_{(I_x)} \in G$. Furthermore, $I_x \subseteq I$, hence $\|x \in \text{Id}(u^{(I_x)})\| \leq \|x \in \text{Id}(u^{(I)})\|$, so, by (8.5), $\|x \in \text{Id}(u^{(I)})\| \in G$, for all $x \in X$. So we have proved that $\|x \in \text{Id}(u^{(I)})\| \wedge \alpha_{(I)} \in G$, for all $x \in X$. By Lemma 7.8, $x/G \in \text{Id}(u^{(I)}/G)$. This holds for all $x \in X$, thus, by (8.3), $a/G \in \text{Id}(u^{(I)}/G)$. By Lemma 7.8, $\|a \in \text{Id}(u^{(I)})\| \wedge \alpha_{(I)} \in G$, whence $g(a) \in G$. This completes the proof of (8.2). \square

Note the following immediate corollary of Proposition 8.10.

Corollary 8.11. *Let P be a D -valued partial lattice, let $f : P \rightarrow D$, let $\alpha \in D$.*

- (i) *Suppose that P has $(\text{Id}\vee)$. If f is an affine lower function, then $(f \wedge \alpha)^{\text{Id}} = f^{\text{Id}} \wedge \alpha$.*
- (ii) *Suppose that P has $(\text{Fil}\vee)$. If f is an affine upper function, then $(f \wedge \alpha)^{\text{Fil}} = f^{\text{Fil}} \wedge \alpha$.*

9. The lattices of affine ideal functions and affine filter functions

In this section, we fix a distributive lattice D with unit, and a D -valued partial lattice P .

Lemma 9.1.

- (i) *Suppose that P has $(\text{Id}\cap)$. Then the meet of any two affine lower functions on P is an affine lower function on P . In particular, $\mathcal{I}_{\text{aff}}(P)$ is closed under meet.*
- (ii) *Suppose that P has $(\text{Fil}\cap)$. Then the meet of any two affine upper functions on P is an affine upper function on P . In particular, $\mathcal{F}_{\text{aff}}(P)$ is closed under meet.*

Proof. We provide a proof for (i). Let $f, g \in \mathcal{I}_{\text{aff}}(P)$. Write

$$f : x \mapsto \bigvee_{i < m} \|x \leq u_i\| \wedge \alpha_i, \quad g : x \mapsto \bigvee_{j < n} \|x \leq v_j\| \wedge \beta_j.$$

By $(\text{Id}\cap)$, there exists a common $(\text{Id}\cap)$ -sample W for all pairs $\{u_i, v_j\}$, for $\langle i, j \rangle \in m \times n$. This means that

$$\|x \leq u_i\| \wedge \|x \leq v_j\| = \bigvee_{w \in W} \|x \leq w\| \wedge \gamma_{i,j,w}, \quad \text{for all } x \in P,$$

where we put $\gamma_{i,j,w} = \|w \leq u_i\| \wedge \|w \leq v_j\|$, for all $\langle i, j, w \rangle \in m \times n \times W$. Hence, for $x \in P$,

$$f(x) \wedge g(x) = \bigvee_{\substack{i < m \\ j < n}} \|x \leq u_i\| \wedge \|x \leq v_j\| \wedge \alpha_i \wedge \beta_j = \bigvee_{w \in W} \|x \leq w\| \wedge \gamma_w,$$

where we put $\gamma_w = \bigvee_{\langle i,j \rangle \in m \times n} \gamma_{i,j,w} \wedge \alpha_i \wedge \beta_j$, for all $w \in W$. Therefore, $f \wedge g$ is an affine lower function.

Since the meet of any two ideal functions on P is an ideal function on P , it follows that $\mathcal{I}_{\text{aff}}(P)$ is closed under meet. \square

Corollary 9.2.

- (i) *If P has both $(\text{Id}\cap)$ and $(\text{Id}\vee)$, then $\mathcal{I}_{\text{aff}}(P)$, with component-wise ordering, is a lattice.*
- (ii) *If P has both $(\text{Fil}\cap)$ and $(\text{Fil}\vee)$, then $\mathcal{F}_{\text{aff}}(P)$, with component-wise ordering, is a lattice.*

Proof. We provide a proof for (i). By Lemma 9.1, $\mathcal{I}_{\text{aff}}(P)$ is closed under meet. If $f, g \in \mathcal{I}_{\text{aff}}(P)$, then $f \vee g$ (the component-wise join of f and g) is an affine lower function on P , thus, by Proposition 8.10, $(f \vee g)^{\text{Id}}$ is defined, and it belongs to $\mathcal{I}_{\text{aff}}(P)$. So $(f \vee g)^{\text{Id}}$ is the join of $\{f, g\}$ in $\mathcal{I}_{\text{aff}}(P)$. \square

In order to differentiate between the component-wise join $f \vee g$ and the join of $\{f, g\}$ in $\mathcal{I}_{\text{aff}}(P)$ (or $\mathcal{F}_{\text{aff}}(P)$), we introduce the following notation.

Notation 9.3. Under the assumptions of Corollary 9.2, we denote by $f \vee^{\text{Id}} g$ (respectively $f \vee^{\text{Fil}} g$) the join of $\{f, g\}$ in $\mathcal{I}_{\text{aff}}(P)$ (respectively in $\mathcal{F}_{\text{aff}}(P)$).

Our next goal is to relate the meet and the join in $\mathcal{I}_{\text{aff}}(P)$ and $\mathcal{F}_{\text{aff}}(P)$ on the one hand, and the meet (intersection) and the join in $\mathcal{I}(P/G)$ and $\mathcal{F}(P/G)$ on the other hand, for a prime filter G of D . For a lower function $f : P \rightarrow D$, the inverse image $f^{-1}G$ of G has the property that if $y \in f^{-1}G$ and $x \leq_G y$ (the preordering \leq_G has been introduced in Section 5), then $x \in f^{-1}G$. Hence, $x/G \in f^{-1}G/G$ iff $f(x) \in G$. This also holds for upper functions on P . Our next result analyzes in more detail the map $f \mapsto f^{-1}G/G$.

Proposition 9.4. Let G be a prime filter of D .

- (i) Suppose that P has $(\text{Id}\cap)$ and $(\text{Id}\vee)$. Then the rule $f \mapsto f^{-1}G/G$ determines a lattice homomorphism from $\langle \mathcal{I}_{\text{aff}}(P), \wedge, \vee^{\text{Id}} \rangle$ to $\langle \mathcal{I}(P/G), \cap, \vee \rangle$.
- (ii) Suppose that P has $(\text{Fil}\cap)$ and $(\text{Fil}\vee)$. Then the rule $f \mapsto f^{-1}G/G$ determines a lattice homomorphism from $\langle \mathcal{F}_{\text{aff}}(P), \wedge, \vee^{\text{Fil}} \rangle$ to $\langle \mathcal{F}(P/G), \cap, \vee \rangle$.

Proof. We provide a proof for (i). We denote by π_G the map $f \mapsto f^{-1}G/G$. If $f : x \mapsto \|x \leq u\| \wedge \alpha$, for fixed $u \in P$ and $\alpha \in D$, then $\pi_G(f)$ equals $\downarrow u/G$ if $\alpha \in G$, \emptyset otherwise, so $\pi_G(f)$ is in both cases an ideal of P/G .

To prove that π_G is a join-homomorphism with range contained in $\text{Id}(P/G)$, it suffices to prove that if $f = \bigvee_{i < n}^{\text{Id}} f_i$, where $n \in \omega \setminus \{0\}$ and $f_i : a \mapsto \|a \leq u_i\| \wedge \alpha_i$ for all $i < n$ (where $u_i \in P$ and $\alpha_i \in D$), then $\pi_G(f)$ is the join of $\{\pi_G(f_i) \mid i < n\}$ in $\text{Id}(P/G)$, that is, we must prove that

$$f^{-1}G/G = \text{Id}\left(\bigcup_{i < n} f_i^{-1}G/G\right). \quad (9.1)$$

So, let $a \in P$. Suppose first that $a/G \in f^{-1}G/G$, that is, $f(a) \in G$. By the formula given for f in Proposition 8.10(i), there exists a nonempty subset I of n such that, using the same notations as in Proposition 8.10(i), $\alpha_{(I)} \in G$ and $\|a \in \text{Id}(u^{(I)})\| \in G$. Therefore, $\alpha_i \in G$ for all $i \in I$, and, by Lemma 7.8, $a/G \in \text{Id}(u^{(I)}/G)$. But, for $i \in I$, $f_i(u_i) = \alpha_i \in G$, thus $u_i/G \in f_i^{-1}G/G$. Therefore, $a/G \in \text{Id}(\bigcup_{i < n} f_i^{-1}G/G)$.

Conversely, suppose that $a/G \in \text{Id}(\bigcup_{i < n} f_i^{-1}G/G)$. We observe that $\bigcup_{i < n} f_i^{-1}G/G$ is generated, as a lower subset of P/G , by $u^{(I)}/G$, where $I = \{i < n \mid \alpha_i \in G\}$. Thus, $a/G \in \text{Id}(u^{(I)}/G)$, so, by Lemma 7.8, $\|a \in \text{Id}(u^{(I)})\| \in G$. Since $\alpha_{(I)} \in G$, $\|a \in \text{Id}(u^{(I)})\| \wedge \alpha_{(I)} \in G$, whence $f(a) \in G$, that is, $a/G \in f^{-1}G/G$.

So we have proved that (9.1) holds. As remarked above, this shows that π_G is a join-homomorphism with range a subset of $\text{Id}(P/G)$.

To conclude the proof, it is sufficient to prove that π_G is a meet-homomorphism. This is easy: for $a \in P$,

$$\begin{aligned} a/G \in \pi_G(f \wedge g) & \quad \text{iff} \quad f(a) \wedge g(a) \in G \\ & \quad \text{iff} \quad f(a) \in G \text{ and } g(a) \in G \\ & \quad \text{iff} \quad a/G \in \pi_G(f) \text{ and } a/G \in \pi_G(g) \\ & \quad \text{iff} \quad a/G \in \pi_G(f) \cap \pi_G(g). \end{aligned}$$

Therefore, $\pi_G(f \wedge g) = \pi_G(f) \cap \pi_G(g)$. \square

10. The elements $\llbracket f \leq g \rrbracket$

We first introduce a convenient notation.

Notation 10.1. Let $\langle \alpha_i \mid i \in I \rangle$ be a family of elements of a lattice D , let $\alpha \in D$. Let $\alpha = \bigvee_{i \in I}^{\text{fin}} \alpha_i$ hold, if there exists a *finite* subset J of I such that $\alpha_i \leq \bigvee_{j \in J} \alpha_j$, for all $i \in I$, and $\alpha = \bigvee_{j \in J} \alpha_j$.

Hence, $\alpha = \bigvee_{i \in I}^{\text{fin}} \alpha_i$ means that the supremum of the α_i is, really, the supremum of a *finite* subfamily of $\langle \alpha_i \mid i \in I \rangle$.

For the remainder of this section, let D be a distributive lattice with unit and let P be a D -valued poset.

Lemma 10.2. Let $m, n \in \omega \setminus \{0\}$, let $u_0, \dots, u_{m-1}, v_0, \dots, v_{n-1} \in P$, $\alpha_0, \dots, \alpha_{m-1}, \beta_0, \dots, \beta_{n-1} \in D$. We define maps f and g from P to D by the rules

$$f(x) = \bigvee_{i < m} \|u_i \leq x\| \wedge \alpha_i, \quad g(x) = \bigvee_{j < n} \|x \leq v_j\| \wedge \beta_j,$$

for all $a \in P$. Put

$$\gamma = \bigvee_{\langle i, j \rangle \in m \times n} \alpha_i \wedge \beta_j \wedge \|u_i \leq v_j\|.$$

Then $\gamma = \bigvee_{x \in P}^{\text{fin}} f(x) \wedge g(x)$.

Proof. For $x \in P$,

$$\begin{aligned} f(x) \wedge g(x) &= \bigvee_{\langle i, j \rangle \in m \times n} \|u_i \leq x\| \wedge \|x \leq v_j\| \wedge \alpha_i \wedge \beta_j \\ &\leq \bigvee_{\langle i, j \rangle \in m \times n} \|u_i \leq v_j\| \wedge \alpha_i \wedge \beta_j \\ &= \gamma. \end{aligned}$$

Conversely, for $\langle i, j \rangle \in m \times n$,

$$\alpha_i \wedge \beta_j \wedge \|u_i \leq v_j\| \leq \alpha_i \wedge g(v_j) \wedge \|u_i \leq v_j\| \leq f(v_j) \wedge g(v_j).$$

The conclusion follows, with $\gamma = \bigvee_{j < n} f(v_j) \wedge g(v_j)$. \square

Definition 10.3. For an affine upper function $f : P \rightarrow D$ and an affine lower function $g : P \rightarrow D$, we put

$$\llbracket f \leq g \rrbracket = \bigvee_{x \in P}^{\text{fin}} f(x) \wedge g(x).$$

By Lemma 10.2, $\llbracket f \leq g \rrbracket$ is always defined, and it is an element of D .

Remark 10.4. With f and g defined as in the statement of Lemma 10.2, we have obtained that

$$\llbracket f \leq g \rrbracket = \bigvee_{j < n} f(v_j) \wedge g(v_j).$$

We could have obtained, similarly, that

$$\llbracket f \leq g \rrbracket = \bigvee_{i < m} f(u_i) \wedge g(u_i).$$

These expressions will be used in Lemma 11.3.

11. Extension of the Boolean values to $\mathbf{W}(P)$

Throughout this section, let D be a distributive lattice with unit, let P be a D -valued partial lattice with $(\text{Id} \cap)$, $(\text{Fil} \cap)$, $(\text{Id} \vee)$, and $(\text{Fil} \vee)$. We shall extend the notation $\|a \leq b\|$, for $a, b \in P$, to all pairs of elements of $\mathbf{W}(P)$.

Definition 11.1. For $\dot{x} \in \mathbf{W}(P)$, we define, by induction on the height of \dot{x} , an affine ideal function \dot{x}^- and an affine filter function \dot{x}^+ on P by the following rules:

- (i) If $\dot{x} = a \in P$, then $\dot{x}^- : t \mapsto \|t \leq a\|$ and $\dot{x}^+ : t \mapsto \|a \leq t\|$.
- (ii) $(\dot{x} \wedge \dot{y})^- = \dot{x}^- \wedge \dot{y}^-$, and $(\dot{x} \vee \dot{y})^- = \dot{x}^- \vee^{\text{Id}} \dot{y}^-$, for all $\dot{x}, \dot{y} \in W(P)$.
- (iii) $(\dot{x} \wedge \dot{y})^+ = \dot{x}^+ \vee^{\text{Fil}} \dot{y}^+$, and $(\dot{x} \vee \dot{y})^+ = \dot{x}^+ \wedge \dot{y}^+$, for all $\dot{x}, \dot{y} \in W(P)$.

We can now provide the D -valued analogue of the notation $\dot{x} \ll \dot{y}$ introduced in Definition 2.5, by using the elements $\llbracket f \leq g \rrbracket$, see Definition 10.3.

Definition 11.2. For $\dot{x}, \dot{y} \in W(P)$, we put

$$\|\dot{x} \ll \dot{y}\| = \llbracket \dot{x}^+ \leq \dot{y}^- \rrbracket.$$

As an easy consequence of Remark 10.4, we record the following lemma.

Lemma 11.3. For $a \in P$ and $\dot{x} \in W(P)$, the following equalities hold:

$$\|a \ll \dot{x}\| = \dot{x}^-(a), \quad \|\dot{x} \ll a\| = \dot{x}^+(a).$$

Definition 11.4. We define $\|\dot{x} \leq \dot{y}\|$, for $\dot{x}, \dot{y} \in W(P)$, by induction on $\max\{\text{ht}(\dot{x}), \text{ht}(\dot{y})\}$, as follows:

- (i) $\|\dot{x} \leq \dot{y}\| = \|\dot{x} \ll \dot{y}\|$, if $\dot{x} \in P$ or $\dot{y} \in P$;
- (ii) $\|\dot{x}_0 \vee \dot{x}_1 \leq \dot{y}_0 \wedge \dot{y}_1\| = \bigwedge_{i,j < 2} \|\dot{x}_i \leq \dot{y}_j\|$;
- (iii) $\|\dot{x}_0 \vee \dot{x}_1 \leq \dot{y}_0 \vee \dot{y}_1\| = \bigwedge_{i < 2} \|\dot{x}_i \leq \dot{y}_0 \vee \dot{y}_1\|$;
- (iv) $\|\dot{x}_0 \wedge \dot{x}_1 \leq \dot{y}_0 \wedge \dot{y}_1\| = \bigwedge_{j < 2} \|\dot{x}_0 \wedge \dot{x}_1 \leq \dot{y}_j\|$;
- (v) $\|\dot{x}_0 \wedge \dot{x}_1 \leq \dot{y}_0 \vee \dot{y}_1\| = \|\dot{x}_0 \wedge \dot{x}_1 \ll \dot{y}_0 \vee \dot{y}_1\| \vee \bigvee_{i,j < 2} \|\dot{x}_i \leq \dot{y}_j\|$.

Proposition 11.5. Let $\dot{x}, \dot{x}_0, \dot{x}_1, \dot{y}, \dot{y}_0, \dot{y}_1 \in W(P)$. Then the following equalities hold:

$$\|\dot{x}_0 \vee \dot{x}_1 \leq \dot{y}\| = \|\dot{x}_0 \leq \dot{y}\| \wedge \|\dot{x}_1 \leq \dot{y}\|; \quad (11.1)$$

$$\|\dot{x} \leq \dot{y}_0 \wedge \dot{y}_1\| = \|\dot{x} \leq \dot{y}_0\| \wedge \|\dot{x} \leq \dot{y}_1\|. \quad (11.2)$$

Proof. By induction on $n < \omega$, we prove that (11.1) (respectively (11.2)) holds for all $\dot{x}_0, \dot{x}_1, \dot{y}$ such that $\text{ht}(\dot{x}_0) + \text{ht}(\dot{x}_1) + \text{ht}(\dot{y}) \leq n$ (respectively for all $\dot{x}, \dot{y}_0, \dot{y}_1$ such that $\text{ht}(\dot{x}) + \text{ht}(\dot{y}_0) + \text{ht}(\dot{y}_1) \leq n$). Let us prove, for example, that (11.1) holds.

Suppose first that $\dot{y} = a \in P$. We compute:

$$\begin{aligned} \|\dot{x}_0 \vee \dot{x}_1 \leq a\| &= \|\dot{x}_0 \vee \dot{x}_1 \ll a\| \quad (\text{by the definition of } \|\dot{x} \leq a\|) \\ &= (\dot{x}_0 \vee \dot{x}_1)^+(a) \quad (\text{by Lemma 11.3}) \\ &= \dot{x}_0^+(a) \wedge \dot{x}_1^+(a) \\ &= \|\dot{x}_0 \ll a\| \wedge \|\dot{x}_1 \ll a\| \\ &= \|\dot{x}_0 \leq a\| \wedge \|\dot{x}_1 \leq a\|. \end{aligned}$$

Next, suppose that $\dot{y} = \dot{y}_0 \wedge \dot{y}_1$. We compute:

$$\begin{aligned}\|\dot{x}_0 \vee \dot{x}_1 \leq \dot{y}\| &= \bigwedge_{i,j < 2} \|\dot{x}_i \leq \dot{y}_j\| \quad (\text{by Definition 11.4(ii)}) \\ &= \|\dot{x}_0 \leq \dot{y}_0 \wedge \dot{y}_1\| \wedge \|\dot{x}_1 \leq \dot{y}_0 \wedge \dot{y}_1\|\end{aligned}$$

(by the induction hypothesis about (11.2))

$$= \|\dot{x}_0 \leq \dot{y}\| \wedge \|\dot{x}_1 \leq \dot{y}\|.$$

Finally, the case $\dot{y} = \dot{y}_0 \vee \dot{y}_1$ is trivial (see Definition 11.4(iii)). \square

For any prime filter G on D , we consider the canonical map

$$\varrho_G : \mathbf{W}(P) \rightarrow \mathbf{W}(P/G), \quad \dot{x} \mapsto \dot{x}/G.$$

For $a \in P$ and for $\dot{x}, \dot{y} \in \mathbf{W}(P)$, ϱ_G satisfies

$$\begin{aligned}\varrho_G(a) &= a/G \in P/G, \\ \varrho_G(\dot{x} \vee \dot{y}) &= \varrho_G(\dot{x}) \vee \varrho_G(\dot{y}), \\ \varrho_G(\dot{x} \wedge \dot{y}) &= \varrho_G(\dot{x}) \wedge \varrho_G(\dot{y}).\end{aligned}$$

We now relate the D -valued \dot{x}^- and \dot{x}^+ with their corresponding classical versions, see Definition 2.4.

Lemma 11.6. *Let G be a prime filter of D . For any $a \in P$ and $\dot{x} \in \mathbf{W}(P)$, the following equivalences hold:*

$$\dot{x}^-(a) \in G \quad \text{iff} \quad a/G \in (\dot{x}/G)^-, \quad (11.3)$$

$$\dot{x}^+(a) \in G \quad \text{iff} \quad a/G \in (\dot{x}/G)^+. \quad (11.4)$$

In other words, $(\dot{x}^-)^{-1}G/G = (\dot{x}/G)^-$ and $(\dot{x}^+)^{-1}G/G = (\dot{x}/G)^+$.

Proof. We provide a proof for (11.3). We argue by induction on the height of \dot{x} . If $\dot{x} = b \in P$, we must prove $\|a \leq b\| \in G$ iff $a/G \in (b/G)^-$, which is the definition of the ordering in P/G .

Suppose that $\dot{x} = \dot{x}_0 \vee \dot{x}_1$. Then $\dot{x}^- = \dot{x}_0^- \vee^{\text{Id}} \dot{x}_1^-$. We compute further:

$$\begin{aligned}(\dot{x}^-)^{-1}G/G &= (\dot{x}_0^-)^{-1}G/G \vee (\dot{x}_1^-)^{-1}G/G \quad \text{in } \mathcal{I}(P/G) \quad (\text{by Proposition 9.4}) \\ &= (\dot{x}_0/G)^- \vee (\dot{x}_1/G)^- \quad (\text{by the induction hypothesis}) \\ &= (\dot{x}_0/G \vee \dot{x}_1/G)^- \quad (\text{see Definition 2.4}) \\ &= ((\dot{x}_0 \vee \dot{x}_1)/G)^- \\ &= (\dot{x}/G)^-.\end{aligned}$$

The proof for the case $\dot{x} = \dot{x}_0 \wedge \dot{x}_1$ is similar. \square

It is now easy to relate the symbols $\dot{x} \ll \dot{y}$ (see Definition 2.5) and $\|\dot{x} \ll \dot{y}\|$.

Corollary 11.7. *Let G be a prime filter of D , let $\dot{x}, \dot{y} \in W(P)$. Then $\dot{x}/G \ll \dot{y}/G$ (in $W(P/G)$) iff $\|\dot{x} \ll \dot{y}\| \in G$.*

We are now ready to extend Corollary 11.7 to the case $\|\dot{x} \leq \dot{y}\|$.

Proposition 11.8. *Let G be a prime filter of D , let $\dot{x}, \dot{y} \in W(P)$. Then $\dot{x}/G \preceq \dot{y}/G$ (in $W(P/G)$) iff $\|\dot{x} \leq \dot{y}\| \in G$.*

Proof. We argue by induction on the pair $(\text{ht}(\dot{x}), \text{ht}(\dot{y}))$. If $\dot{x} \in P$ or $\dot{y} \in P$, the conclusion follows from Corollary 11.7. The cases $\dot{x} = \dot{x}_0 \vee \dot{x}_1$ and $\dot{y} = \dot{y}_0 \wedge \dot{y}_1$, $\dot{x} = \dot{x}_0 \vee \dot{x}_1$ and $\dot{y} = \dot{y}_0 \vee \dot{y}_1$, and $\dot{x} = \dot{x}_0 \wedge \dot{x}_1$ and $\dot{y} = \dot{y}_0 \wedge \dot{y}_1$ are obvious by the induction hypothesis. The case $\dot{x} = \dot{x}_0 \wedge \dot{x}_1$ and $\dot{y} = \dot{y}_0 \vee \dot{y}_1$ is easy by Corollary 11.7 and the induction hypothesis. \square

By using the fact that the relation \preceq (see Definition 2.6) is reflexive and transitive in each P/G (see Lemma 2.7) and by Lemma 4.6, we obtain the following consequence.

Corollary 11.9. *For $\dot{x}, \dot{y}, \dot{z} \in W(P)$, the following inequalities hold:*

$$\|\dot{x} \leq \dot{x}\| = 1; \quad \|\dot{x} \leq \dot{y}\| \wedge \|\dot{y} \leq \dot{z}\| \leq \|\dot{x} \leq \dot{z}\|.$$

Remark 11.10. Of course, the identity $\|\dot{x} \leq \dot{x}\| = 1$ is easy to prove directly. However, proving the inequality $\|\dot{x} \leq \dot{y}\| \wedge \|\dot{y} \leq \dot{z}\| \leq \|\dot{x} \leq \dot{z}\|$ directly is much less intuitive (though, of course, possible) if one has to avoid the use of prime filters of D .

In particular, $W(P)$ is a D -valued poset.

Part 3. D -comeasured partial lattices

12. Finitely covered D -comeasured partial lattices

Our next definition will be a combination between the definition of a D -valued poset and a partial lattice. In this section, we fix a distributive lattice D with unit.

Definition 12.1. A D -comeasured partial lattice is a structure $\langle P, \|_ _ \|, \leq, \vee, \wedge \rangle$ that satisfies the following axioms:

- (i) $\langle P, \leq, \vee, \wedge \rangle$ is a partial lattice.
- (ii) $\langle P, \|_ _ \| \rangle$ is a D -valued poset.
- (iii) $x \leq y$ implies that $\|x \leq y\| = 1$.

- (iv) $a = \bigvee X$ implies that $\|a \leq b\| = \bigwedge_{x \in X} \|x \leq b\|$, for all $a, b \in P$ and all $X \in \text{dom } \bigvee$.
 (v) $a = \bigwedge X$ implies that $\|b \leq a\| = \bigwedge_{x \in X} \|b \leq x\|$, for all $a, b \in P$ and all $X \in \text{dom } \bigwedge$.

Definition 12.2. Let P be a D -valued poset, let \mathcal{D} be a subset of $[P]_*^{<\omega}$. If $X \in [P]_*^{<\omega}$, then a \mathcal{D} -cover of X is a *nonempty finite* subset \mathcal{D}' of \mathcal{D} such that

$$\|X = Y\| \leq \bigvee_{Z \in \mathcal{D}'} \|X = Z\|, \quad \text{for all } Y \in \mathcal{D}.$$

We say that \mathcal{D} is *finitely covering*, if every element of $[P]_*^{<\omega}$ has a \mathcal{D} -cover.

We say that a D -comeasured partial lattice P is *finitely covering*, if both $\text{dom } \bigvee$ and $\text{dom } \bigwedge$ are finitely covering subsets of $[P]_*^{<\omega}$.

Observe that if \mathcal{D}' is a \mathcal{D} -cover of X , then any finite subset of \mathcal{D} that contains \mathcal{D}' is also a \mathcal{D} -cover of X .

Observe also that in the definition of a \mathcal{D} -cover, we could have replaced the inequality

$$\|X = Y\| \leq \bigvee_{Z \in \mathcal{D}'} \|X = Z\| \quad \text{by} \quad \|X = Y\| \leq \bigvee_{Z \in \mathcal{D}'} \|Y = Z\|,$$

since $\|X = Y\| \wedge \|X = Z\| = \|X = Y\| \wedge \|Y = Z\|$ (by Lemma 4.3).

Lemma 12.3. Let P be a D -valued poset, let \mathcal{D} be a finitely covering subset of $[P]_*^{<\omega}$. Then for all $X \in [P]_*^{<\omega}$, there exists a nonempty finite subset \mathcal{D}' of \mathcal{D} such that

$$\|Y \subseteq X\| \leq \bigvee_{Z \in \mathcal{D}'} \|Y = Z\|, \quad \text{for all } Y \in \mathcal{D}.$$

Proof. Pick a common \mathcal{D} -cover, \mathcal{D}' , of all nonempty subsets of X . We compute:

$$\begin{aligned} \|Y \subseteq X\| &= \bigvee_{\emptyset \subset T \subseteq X} \|Y = T\| \quad (\text{by Lemma 4.5}) \\ &= \bigvee_{\emptyset \subset T \subseteq X} \bigvee_{Z \in \mathcal{D}'} \|Y = T\| \wedge \|T = Z\| \quad (\text{because } Y \in \mathcal{D}) \\ &\leq \bigvee_{Z \in \mathcal{D}'} \|Y = Z\| \quad \text{by Lemma 4.3.} \quad \square \end{aligned}$$

We observe that the condition that P be finitely covering, for a D -comeasured partial lattice P , implies that both $\text{dom } \bigvee$ and $\text{dom } \bigwedge$ are nonempty.

The following observation, although trivial, provides us with two important classes of finitely covering D -comeasured partial lattices.

Proposition 12.4. Let P be a D -comeasured partial lattice. Each of the following conditions implies that P is finitely covering:

- (i) P is finite, and both $\text{dom } \bigvee$ and $\text{dom } \bigwedge$ are nonempty.
- (ii) P is a lattice, that is, $\text{dom } \bigvee = \text{dom } \bigwedge = [P]_*^{<\omega}$.

Proof. (i) It is obvious that for every nonempty subset \mathcal{D} of $[P]_*^{<\omega}$, \mathcal{D} is a \mathcal{D} -cover of every element of $[P]_*^{<\omega}$. This holds, in particular, for $\text{dom } \bigvee$ and $\text{dom } \bigwedge$.

- (ii) For $X \in [P]_*^{<\omega}$, $\{X\}$ is simultaneously a $\text{dom } \bigvee$ -cover and a $\text{dom } \bigwedge$ -cover of X . \square

Now we state the fundamental connection between D -comeasured partial lattices and D -valued partial lattices:

Proposition 12.5. *Let P be a finitely covering D -comeasured partial lattice. Then P extends to a D -valued partial lattice \tilde{P} that satisfies*

$$\|a = \bigvee X\| = \bigvee^{\text{fin}} \left\langle \|a = b\| \wedge \|X = Y\| \mid \langle b, Y \rangle \in P \times [P]_*^{<\omega}, b = \bigvee Y \right\rangle, \quad (12.1)$$

$$\|a = \bigwedge X\| = \bigvee^{\text{fin}} \left\langle \|a = b\| \wedge \|X = Y\| \mid \langle b, Y \rangle \in P \times [P]_*^{<\omega}, b = \bigwedge Y \right\rangle, \quad (12.2)$$

for all $\langle a, X \rangle \in P \times [P]_*^{<\omega}$. Furthermore, \tilde{P} is finitely sampled.

Proof. We first prove a claim.

Claim 1.

- (i) $\|X \subseteq Y\| \leq \|a \leq b\|$, for all $a, b \in P$ and all $X, Y \in [P]_*^{<\omega}$ such that $a = \bigvee X$ and $b = \bigvee Y$.
- (i*) $\|X \subseteq Y\| \leq \|b \leq a\|$, for all $a, b \in P$ and all $X, Y \in [P]_*^{<\omega}$ such that $a = \bigwedge X$ and $b = \bigwedge Y$.
- (ii) $\|X = Y\| \leq \|a = b\|$, for all $a, b \in P$ and all $X, Y \in [P]_*^{<\omega}$ such that $a = \bigvee X$ and $b = \bigvee Y$.
- (ii*) $\|X = Y\| \leq \|a = b\|$, for all $a, b \in P$ and all $X, Y \in [P]_*^{<\omega}$ such that $a = \bigwedge X$ and $b = \bigwedge Y$.

Proof of Claim 1. We first prove (i). Let $x \in X$. The inequality $y \leq b$ holds for all $y \in Y$, thus $\|y \leq b\| = 1$, so

$$\|x = y\| = \|x = y\| \wedge \|y \leq b\| \leq \|x \leq b\|.$$

Hence,

$$\|X \subseteq Y\| \leq \|x \in Y\| = \bigvee_{y \in Y} \|x = y\| \leq \|x \leq b\|,$$

so, since $a = \bigvee X$,

$$\|X \subseteq Y\| \leq \bigwedge_{x \in X} \|x \leq b\| = \|a \leq b\|.$$

(i*) is dual of (i), and (ii), (ii*) follow immediately. \square

We prove now that Eqs. (12.1) and (12.2) are consistent definitions of $\|a = \bigvee X\|$ and $\|a = \bigwedge X\|$. We do it, for example, for (12.1). So let $\langle a, X \rangle \in P \times [P]_*^{<\omega}$. We put

$$\alpha = \bigvee_{i < n} \|a = a_i\| \wedge \|X = X_i\|,$$

where $\{X_i \mid i < n\}$ is a $\text{dom } \bigvee$ -cover of X and $a_i = \bigvee X_i$, for all $i < n$. For $\langle b, Y \rangle \in P \times [P]_*^{<\omega}$ such that $b = \bigvee Y$, we compute:

$$\begin{aligned} \|a = b\| \wedge \|X = Y\| &= \bigvee_{i < n} \|a = b\| \wedge \|X = Y\| \wedge \|X = X_i\| \quad (\text{by the definition of a dom } \bigvee\text{-cover}) \\ &= \bigvee_{i < n} \|a = b\| \wedge \|X_i = Y\| \wedge \|X = X_i\| \quad (\text{by an easy application of Lemma 4.3}) \\ &\leq \bigvee_{i < n} \|a = b\| \wedge \|a_i = b\| \wedge \|X = X_i\| \quad (\text{by Claim 1 applied to } \langle X_i, Y \rangle) \\ &\leq \bigvee_{i < n} \|a = a_i\| \wedge \|X = X_i\| = \alpha, \end{aligned}$$

hence, $\alpha = \bigvee^{\text{fin}} \langle \|a = b\| \wedge \|X = Y\| \mid \langle b, Y \rangle \in P \times [P]_*^{<\omega}, b = \bigvee Y \rangle$. This settles (12.1). The proof for (12.2) is dual.

We now verify that all items of Definition 5.1 are satisfied by the Boolean values obtained above.

Condition 1. $\|a = \bigvee X\| \wedge \|a \leq b\| = \|a = \bigvee X\| \wedge \bigwedge_{x \in X} \|x \leq b\|$.

Let $\{X_i \mid i < n\}$ be a $\text{dom } \bigvee$ -cover of X . Put $a_i = \bigvee X_i$, for all $i < n$. We compute:

$$\begin{aligned} \|a = \bigvee X\| \wedge \|a \leq b\| &= \bigvee_{i < n} \|a = a_i\| \wedge \|X = X_i\| \wedge \|a \leq b\| \\ &= \bigvee_{i < n} \|a = a_i\| \wedge \|X = X_i\| \wedge \|a_i \leq b\| \end{aligned}$$

(because $\|a = a_i\| \wedge \|a \leq b\| = \|a = a_i\| \wedge \|a_i \leq b\|$)

$$= \bigvee_{i < n} \left(\|a = a_i\| \wedge \|X = X_i\| \wedge \bigwedge_{x \in X_i} \|x \leq b\| \right)$$

(because $a_i = \bigvee X_i$)

$$= \bigvee_{i < n} \left(\|a = a_i\| \wedge \|X = X_i\| \wedge \bigwedge_{x \in X} \|x \leq b\| \right)$$

(by Lemma 4.4)

$$= \|a = \bigvee X\| \wedge \bigwedge_{x \in X} \|x \leq b\|.$$

Condition 2. $\|a = \bigvee X\| \wedge \|X = Y\| \leq \|a = \bigvee Y\|.$

Again, let $\{X_i \mid i < n\}$ be a common dom \bigvee -cover of X and Y , and put $a_i = \bigvee X_i$, for all $i < n$. We compute:

$$\begin{aligned} \|a = \bigvee X\| \wedge \|X = Y\| &= \bigvee_{i < n} \|a = a_i\| \wedge \|X = X_i\| \wedge \|X = Y\| \\ &\leq \bigvee_{i < n} \|a = a_i\| \wedge \|Y = X_i\| \quad (\text{by Lemma 4.3}) \\ &\leq \|a = \bigvee Y\|. \end{aligned}$$

Condition 3. $\|a = \bigvee X\| \wedge \|a = b\| \leq \|b = \bigvee X\|.$

Again, let $\{X_i \mid i < n\}$ be a dom \bigvee -cover of X , and put $a_i = \bigvee X_i$, for all $i < n$. We compute:

$$\begin{aligned} \|a = \bigvee X\| \wedge \|a = b\| &= \bigvee_{i < n} \|a = a_i\| \wedge \|X = X_i\| \wedge \|a = b\| \\ &\leq \bigvee_{i < n} \|b = a_i\| \wedge \|X = X_i\| \\ &= \|b = \bigvee X\|. \end{aligned}$$

Hence we have verified items (1)–(3) of Definition 5.1. The items (1*), (2*), and (3*) are dual.

At this point, we have verified that (12.1) and (12.2) define a structure of D -valued partial lattice \tilde{P} on P .

It remains to prove that \tilde{P} is finitely sampled. We verify, for example, that \tilde{P} is finitely join-sampled. So, let $X \in [P]_*^{<\omega}$. Let $\{X_i \mid i < n\}$ be a dom \bigvee -cover of X , and put $a_i = \bigvee X_i$, for all $i < n$. Put $U = \{a_i \mid i < n\}$. For $a \in P$ and $i < n$,

$$\|a = a_i\| \wedge \|X = X_i\| \leq \|a_i = a_i\| \wedge \|X = X_i\| \leq \|a_i = \bigvee X\|,$$

so we obtain the following inequalities:

$$\|a = \bigvee X\| = \bigvee_{i < n} \|a = a_i\| \wedge \|X = X_i\| \leq \bigvee_{i < n} \|a_i = \bigvee X\| = \bigvee_{u \in U} \|u = \bigvee X\|.$$

Therefore, U is a join-sample of X . Dually, \tilde{P} is finitely meet-sampled. \square

Remark 12.6. In particular, in the context of Proposition 12.5, the notation $\|a = \bigvee X\|$, for $\bigvee X$ defined, is unambiguous, because the singleton $\{X\}$ is a $\text{dom } \bigvee$ -cover of X , thus, if b is defined as $\bigvee X$, then $\|a = \bigvee X\|$ (as defined in (12.1)) equals $\|a = b\|$. Of course, the dual statement holds for the meet.

We can now state a useful strengthening of Proposition 12.4.

Proposition 12.7. *Let P be a D -comeasured partial lattice. Each of the following conditions implies that P is finitely covering and that the associated D -valued partial lattice satisfies $(\text{Id}\cap)$, $(\text{Id}\vee)$, $(\text{Fil}\cap)$, and $(\text{Fil}\vee)$:*

- (i) P is finite and both $\text{dom } \bigvee$ and $\text{dom } \bigwedge$ are nonempty.
- (ii) P is a lattice.

Proof. In both cases, it follows from Proposition 12.4 that P is finitely covering. We denote by \tilde{P} the associated finitely sampled D -comeasured partial lattice, see Proposition 12.5.

(i) We assume that P is finite. For $X \in [P]_*^{<\omega}$, it is obvious that P is a finite $(\text{Id}\cap)$ -sample of X . Hence, \tilde{P} has $(\text{Id}\cap)$.

We prove that P is also an $(\text{Id}\vee)$ -sample of X . Indeed, if G is a prime filter of D and if $k < \omega$, then the condition

$$\text{Id}_0(X/G, P/G) \subset \text{Id}_1(X/G, P/G) \subset \cdots \subset \text{Id}_k(X/G, P/G)$$

implies that $k < |P/G|$, thus, a fortiori, $k < |P|$. In particular, $\text{Id}_{|P|-1}(X/G, P/G) = \text{Id}_{|P|}(X/G, P/G)$. Hence, by Lemma 7.9, P is an $(\text{Id}\vee)$ -sample of X , with index $|P| - 1$. So \tilde{P} has $(\text{Id}\vee)$. The dual statements, about $(\text{Fil}\cap)$ and $(\text{Fil}\vee)$, are proved similarly.

(ii) Assume that P is a lattice. For $X \in [P]_*^{<\omega}$, if we put $a = \bigwedge X$, then $\|a = \bigwedge X\| = 1$; thus $\{a\}$ is an $(\text{Id}\cap)$ -sample of X . Put $b = \bigvee X$, and let U be a finite subset of P containing X . For every $x \in P$ and every Z such that $\emptyset \subset Z \subseteq U$, we have that $\|Z \subseteq \text{Id}_0(X, U)\| \leq \bigwedge_{z \in Z} \|z \leq b\| = \|\bigvee Z \leq b\|$, from which it follows that $\|x \leq \bigvee Z\| \wedge \|Z \subseteq \text{Id}_0(X, U)\| \leq \|x \leq b\|$. Since the value $\|x \leq b\|$ is reached for $Z = X$, we conclude that $\|x \in \text{Id}_1(X, U)\| = \|x \leq b\|$. Now we observe that $x \mapsto \|x \leq b\|$ is an ideal function on \tilde{P} . It follows that X is an $(\text{Id}\vee)$ -sample of X , with index 1. There are similar, dual statements, for $(\text{Fil}\cap)$ and $(\text{Fil}\vee)$. \square

13. Statement and proof of Theorem A

In order to relate D -comeasured partial lattices and congruence lattices of partial lattices, we state the following simple result.

Proposition 13.1. *Let D be a distributive lattice with unit, let $\langle P, \leq, \bigvee, \bigwedge \rangle$ be a partial lattice, endowed with a map $P \times P \rightarrow D$, $\langle x, y \rangle \mapsto \|x \leq y\|$. Then the following are equivalent:*

- (i) $\langle P, \|-\|, \leq, \bigvee, \bigwedge \rangle$ is a D -comeasured partial lattice.
(ii) There exists a homomorphism $\varphi: \langle \text{Con}_c P, \vee, \mathbf{0}_P \rangle \rightarrow \langle D, \wedge, 1 \rangle$ such that

$$\|x \leq y\| = \varphi(\Theta_P^+(x, y)), \quad \text{for all } x, y \in P.$$

Proof. (i) \Rightarrow (ii). It is sufficient to prove that if $n < \omega$ and $a, b, a_0, \dots, a_{n-1}, b_0, \dots, b_{n-1} \in P$, the condition

$$\Theta_P^+(a, b) \subseteq \bigvee_{i < n} \Theta_P^+(a_i, b_i) \quad (13.1)$$

implies that

$$\|a \leq b\| \supseteq \bigwedge_{i < n} \|a_i \leq b_i\|. \quad (13.2)$$

So put $\alpha = \bigwedge_{i < n} \|a_i \leq b_i\|$ and define a binary relation \leq_θ on P by the rule

$$x \leq_\theta y \quad \text{iff} \quad \alpha \leq \|x \leq y\|, \quad \text{for all } x, y \in P.$$

We verify that θ is a congruence of the partial lattice P . It is obvious that θ is a pre-ordering of P , see Definition 4.1, and that θ contains the ordering \leq of P . Let $u \in P$, let $X \in [P]_*^{<\omega}$, we verify that $u = \bigvee X$ (respectively $u = \bigwedge X$) implies that u is the supremum (respectively the infimum) of X with respect to θ . We do it for example for the join. From $X \leq u$, it follows that $X \leq_\theta u$. Now let $v \in P$ such that $X \leq_\theta v$, that is, $\alpha \leq \|x \leq v\|$, for all $x \in X$. Since $u = \bigvee X$, it follows from Definition 12.1 that $\alpha \leq \|u \leq v\|$; that is, $u \leq_\theta v$. This proves our assertion about the supremum. The proof for the infimum is dual.

Now, by the definition of α , the inequality $a_i \leq_\theta b_i$ holds for all $i < n$, that is, $\Theta_P^+(a_i, b_i) \subseteq \theta$. Hence, by (13.1), $\Theta_P^+(a, b) \subseteq \theta$; that is, $\alpha \leq \|a \leq b\|$; in other words, (13.2) holds.

(ii) \Rightarrow (i). This direction of the proof follows immediately from the identities

$$\Theta_P^+(a, b) = \begin{cases} \bigvee_{x \in X} \Theta_P^+(x, b), & \text{if } a = \bigvee X, \\ \bigvee_{x \in X} \Theta_P^+(b, x), & \text{if } a = \bigwedge X, \end{cases}$$

for all $a, b \in P$ and all $X \in [P]_*^{<\omega}$. \square

Definition 13.2. Let D be a distributive lattice with zero. A D -measured partial lattice is a pair $\langle P, \varphi \rangle$, where P is a partial lattice and $\varphi: \text{Con}_c P \rightarrow D$ is a $\{\vee, 0\}$ -homomorphism.

If, in addition, P is a lattice, we say that $\langle P, \varphi \rangle$ is a D -measured lattice.

We say that $\langle P, \varphi \rangle$ is *proper*, if φ isolates 0 (that is, $\varphi(\theta) = 0$ iff $\theta = \mathbf{0}_P$, for all θ in $\text{Con}_c P$).

Hence, by Proposition 13.1, the notions of a D -comeasured partial lattice and a D -measured partial lattice are, up to dualization of D , *equivalent*.

Definition 13.3. Let D be a distributive lattice with unit and let P be a *finitely covering* D -comeasured partial lattice. We say that P has $(\text{Id}\cap)$ (respectively $(\text{Id}\vee)$, $(\text{Fil}\cap)$, $(\text{Fil}\vee)$), if the D -valued partial lattice \tilde{P} of Proposition 12.5 has $(\text{Id}\cap)$ (respectively $(\text{Id}\vee)$, $(\text{Fil}\cap)$, $(\text{Fil}\vee)$).

We say that P is *balanced*, if it has $(\text{Id}\cap)$, $(\text{Id}\vee)$, $(\text{Fil}\cap)$, and $(\text{Fil}\vee)$.

If D is a distributive lattice with zero, we say that a D -measured partial lattice P is *balanced*, if the associated D^d -comeasured partial lattice is balanced. Similar definitions hold for $(\text{Id}\cap)$, $(\text{Id}\vee)$, $(\text{Fil}\cap)$, and $(\text{Fil}\vee)$.

By Proposition 12.7, every D -measured lattice, or every finite D -measured partial lattice with nonempty meet and join, is balanced.

Now we can provide a precise statement, and proof, for Theorem A. We recall that j_P is the natural embedding from P into $\mathbf{F}_L(P)$, see Section 2.2.

Theorem A. Let D be a distributive lattice with zero, let $\langle P, \varphi \rangle$ be a balanced D -measured partial lattice. Then there exists a $\{\vee, 0\}$ -homomorphism $\psi : \text{Con}_c \mathbf{F}_L(P) \rightarrow D$ such that

$$\psi \circ \text{Con}_c j_P = \varphi.$$

The remainder of this section will be devoted to the proof of Theorem A.

We first endow P with its natural structure of D^d -co measured partial lattice, see Proposition 13.1. By assumption, this structure is balanced, that is, it is finitely covering and it has $(\text{Id}\cap)$, $(\text{Id}\vee)$, $(\text{Fil}\cap)$, and $(\text{Fil}\vee)$, see Definition 13.3. We define elements $\|\dot{x} \leq \dot{y}\|$, for all elements \dot{x}, \dot{y} of $W(P)$, as in Definition 11.4. We define binary relations \leq^* and \equiv^* on $W(P)$ by the rules

$$\begin{aligned} \dot{x} \leq^* \dot{y} & \text{ iff } \|\dot{x} \leq \dot{y}\| = 1, \\ \dot{x} \equiv^* \dot{y} & \text{ iff } \|\dot{x} = \dot{y}\| = 1. \end{aligned}$$

It follows from Corollary 11.9 that \leq^* is a preordering of $W(P)$ and that \equiv^* is the associated equivalence relation. Let $L = \langle W(P), \leq^* \rangle / \equiv^*$ be the quotient poset. For $\dot{x} \in W(P)$, we denote by $[\dot{x}]$ the \equiv^* -equivalence class of \dot{x} . For $x, y \in L$ and $\dot{x} \in x, \dot{y} \in y$, the element $\|\dot{x} \leq \dot{y}\|$ does not depend of the choice of $\langle \dot{x}, \dot{y} \rangle$, we denote it by $\|x \leq y\|$. Similarly, we define $\|x = y\| = \|\dot{x} = \dot{y}\|$. Furthermore, it follows from Proposition 11.5 that $[\dot{x} \vee \dot{y}]$ (respectively $[\dot{x} \wedge \dot{y}]$) is the supremum (respectively the infimum) of $\{x, y\}$ in L .

Hence, L is a lattice. Furthermore, by Proposition 11.5, the equality

$$\|x_0 \vee x_1 \leq y\| = \|x_0 \leq y\| \wedge \|x_1 \leq y\|$$

holds for all $x_0, x_1, y \in L$. Symmetrically, the equality

$$\|x \leq y_0 \wedge y_1\| = \|x \leq y_0\| \wedge \|x \leq y_1\|$$

holds for all $x, y_0, y_1 \in L$. An easy induction shows that $\|\bigvee X \leq a\| = \bigwedge_{x \in X} \|x \leq a\|$ and $\|a \leq \bigwedge X\| = \bigwedge_{x \in X} \|a \leq x\|$, for all $a \in L$ and all $X \in [L]_*^{<\omega}$. Hence, L is a D^d -co-measured partial lattice, see Definition 12.1. Therefore, by Proposition 13.1, there exists a $\{\vee, 0\}$ -homomorphism $\rho : \text{Con}_c L \rightarrow D$ such that

$$\rho(\Theta_L^+(x, y)) = \|x \leq y\|, \quad \text{for all } x, y \in L. \quad (13.3)$$

Furthermore, it is easy to verify that the rule $a \mapsto [a]$ defines a homomorphism of partial lattices from P to L . Thus, since L is a lattice, there exists, by Proposition 2.8, a unique lattice homomorphism $f : \mathbf{F}_L(P) \rightarrow L$ such that $f(a) = [a]$, for all $a \in P$.

We put $\psi = \rho \circ \text{Con}_c f$, a $\{\vee, 0\}$ -homomorphism from $\text{Con}_c \mathbf{F}_L(P)$ to D . For $a, b \in P$,

$$\psi(\Theta_{\mathbf{F}_L(P)}^+(a, b)) = \rho(\Theta_L^+([a], [b])) = \|[a] \leq [b]\| = \|a \leq b\| = \varphi(\Theta_P^+(a, b)),$$

so $\psi \circ \text{Con}_c j_P = \varphi$. This concludes the proof of Theorem A.

14. Quotients of D -comeasured partial lattices by prime filters

In this section, we fix a distributive lattice D with unit.

If P is a finitely covering D -comeasured partial lattice, then, by Proposition 12.5, P extends canonically to a D -valued partial lattice. So, by using Proposition 5.6, we can define a partial lattice P/G , for every prime filter G of D . Our next result describes the join and meet operations in P/G .

Lemma 14.1. *Let P be a finitely covering D -comeasured partial lattice, let $a \in P/G$, let $X \in [P/G]_*^{<\omega}$. Then the following assertions hold:*

- (i) $a = \bigvee X$ in P/G iff there are $a \in P$ and $X \in [P]_*^{<\omega}$ such that $a = a/G$, $X = X/G$, and $a = \bigvee X$.
- (ii) $a = \bigwedge X$ in P/G iff there are $a \in P$ and $X \in [P]_*^{<\omega}$ such that $a = a/G$, $X = X/G$, and $a = \bigwedge X$.

Proof. We prove (i); (ii) is dual. Suppose first that $a = \bigvee X$. Pick $a \in a$ and $X \in [P]_*^{<\omega}$ such that $X = X/G$. By the definition of the join operation in P/G , $\|a = \bigvee X\| \in G$. Let $\{X_i \mid i < n\}$ (where $n > 0$) be a dom \bigvee -cover of X . Put $a_i = \bigvee X_i$, for all $i < n$. So the equality

$$\|a = \bigvee X\| = \bigvee_{i < n} \|a = a_i\| \wedge \|X = X_i\|$$

holds by definition, thus, since G is prime, there exists $i < n$ such that $a = a_i/G$ and $X = X_i/G$. Since $a_i = \bigvee X_i$, we have proved the “if” direction of the implication in (i).

To prove the converse, assume that $a = a/G$, $X = X/G$, and $a = \bigvee X$. Since $\{X\}$ is a dom \bigvee -cover of X , $\|a = \bigvee X\| = 1 \in G$, hence $a/G = \bigvee(X/G)$ in P/G by the definition of \bigvee in P/G , so $a = \bigvee X$. \square

Definition 14.2. Let P and Q be D -comeasured partial lattices, let $f : P \rightarrow Q$ be a homomorphism of partial lattices. We say that f is

- (i) a *uniform map*, if $\|x \leq y\| \leq \|f(x) \leq f(y)\|$, for all $x, y \in P$;
- (ii) an *isometry*, if f is an embedding of partial lattices and $\|x \leq y\| = \|f(x) \leq f(y)\|$, for all $x, y \in P$.

Lemma 14.3. Let P and Q be finitely covering D -comeasured partial lattices and let $f : P \rightarrow Q$ be a uniform map. For any prime filter G of D , one can define a homomorphism $f^G : P/G \rightarrow Q/G$ of partial lattices by the rule

$$f^G(x/G) = f(x)/G, \quad \text{for all } x \in P. \quad (14.1)$$

Furthermore, if f is an isometry, then f^G is an embedding of partial lattices.

Proof. For $x, y \in P$, $\|x \leq y\| \in G$ implies that $\|f(x) \leq f(y)\| \in G$; so, (14.1) defines a unique order-preserving map $f^G : P/G \rightarrow Q/G$.

We prove that f^G is a homomorphism of partial lattices. We do it for example for the join. So let $\mathbf{a} \in P/G$, $\mathbf{X} \in [P/G]_*^{\leq \omega}$ such that $\mathbf{a} = \bigvee \mathbf{X}$. By Lemma 14.1, there are $a \in P$ and $X \in [P]_*^{\leq \omega}$ such that $\mathbf{a} = a/G$, $\mathbf{X} = X/G$, and $a = \bigvee X$. Since f is a homomorphism of partial lattices, $f(a) = \bigvee f[X]$. Thus, again by Lemma 14.1, $f(a)/G = \bigvee f[X]/G$, that is, $f^G(\mathbf{a}) = \bigvee f^G[\mathbf{X}]$. The proof for the meet is dual.

Finally, if f is an isometry, then $\|x \leq y\| \in G$ iff $\|f(x) \leq f(y)\| \in G$, for all $x, y \in P$, thus f^G is an order-embedding. \square

15. Amalgamation of D -comeasured partial lattices above a finite lattice

We extend in this section the results of Section 3 to D -comeasured partial lattices. We fix a distributive lattice D with unit.

Definition 15.1. A *V-formation* of D -comeasured partial lattices is a structure $\langle K, P, Q, f, g \rangle$ subject to the following conditions:

- (DV1) K, P, Q are D -comeasured partial lattices.
- (DV2) $f : K \hookrightarrow P$ and $g : K \hookrightarrow Q$ are *isometries*.

A V-formation $\langle K, P, Q, f, g \rangle$ is *standard*, if the following conditions hold:

- (SDV1) K is a *finite lattice*.
- (SDV2) $K = P \cap Q$ (set-theoretically), and f and g are, respectively, the inclusion map from K into P and the inclusion map from K into Q .

Remark 15.2. The definition of a standard V-formation of D -comeasured partial lattices is *not* a generalization of the definition of a standard V-formation of partial lattices (Definition 3.1): indeed, observe the additional requirement that K be *finite*.

As in Section 3, we shall write $\langle K, P, Q \rangle$ instead of $\langle K, P, Q, f, g \rangle$ for standard V-formation. The following analogue of Lemma 3.2 trivially holds.

Lemma 15.3. *Every V-formation $\langle K, P, Q, f, g \rangle$ of D -comeasured partial lattices, with K a finite lattice, is isomorphic to a standard V-formation.*

The definition of an amalgam or a pushout of a V-formation is, *mutatis mutandis*, exactly the same as in Definition 3.3. The corresponding analogue of Proposition 3.4 is then the following.

Proposition 15.4. *Let $\mathcal{D} = \langle K, P, Q, f, g \rangle$ be a V-formation of D -comeasured partial lattices, with K a finite lattice. Then \mathcal{D} has a pushout in the category of D -comeasured partial lattices and uniform maps.*

Furthermore, assume that \mathcal{D} is a standard V-formation. Then the pushout $\langle R, f', g' \rangle$ of \mathcal{D} can be described by the following data:

- (a) $R = P \amalg_K Q$ as a partial lattice (see Notation 3.5). In particular, $R = P \cup Q$ set-theoretically.
- (b) For $x, y \in R$, the Boolean value $\|x \leq y\|$ can be computed as follows:
 - (b1) $\|x \leq y\| = \|x \leq y\|_P$, if $x, y \in P$;
 - (b2) $\|x \leq y\| = \|x \leq y\|_Q$, if $x, y \in Q$;
 - (b3) $\|x \leq y\| = \bigvee_{z \in K} \|x \leq z\|_P \wedge \|z \leq y\|_Q$, if $x \in P$ and $y \in Q$;
 - (b4) $\|x \leq y\| = \bigvee_{z \in K} \|x \leq z\|_Q \wedge \|z \leq y\|_P$, if $x \in Q$ and $y \in P$.

Furthermore, both f' and g' are isometries.

Proof. We first prove the mutual compatibility of (b1)–(b4) above. Up to symmetry between P and Q , this amounts to considering the three following cases.

Case 1. $x, y \in K$, prove that $\|x \leq y\|_P = \|x \leq y\|_Q$. This follows immediately from the fact that both maps $f: K \hookrightarrow P$ and $g: K \hookrightarrow Q$ are isometries, thus $\|x \leq y\|_P = \|x \leq y\|_Q = \|x \leq y\|_K$.

Case 2. $x \in P, y \in K$, prove that

$$\|x \leq y\|_P = \bigvee_{z \in K} \|x \leq z\|_P \wedge \|z \leq y\|_Q.$$

For $z \in K$,

$$\|x \leq z\|_P \wedge \|z \leq y\|_Q = \|x \leq z\|_P \wedge \|z \leq y\|_P \quad (\text{because } z, y \in K)$$

$$\leq \|x \leq y\|_P,$$

and, for $z = y$, $\|x \leq z\|_P \wedge \|z \leq y\|_Q = \|x \leq y\|_P$, which proves our assertion.

Case 3. $x, y \in K$, prove that

$$\bigvee_{z \in K} \|x \leq z\|_P \wedge \|z \leq y\|_Q = \bigvee_{z \in K} \|x \leq z\|_Q \wedge \|z \leq y\|_P.$$

By Case 2, the left-hand side and the right-hand side of the equality above are both equal to $\|x \leq y\|_P$ (and to $\|x \leq y\|_Q$).

Now we verify that $\langle x, y \rangle \mapsto \|x \leq y\|$ defines a structure of D -valued poset on R . It is obvious that $\|a \leq a\| = 1$, for all $a \in R$.

Now let $a, b, c \in R$, we prove the inequality

$$\|a \leq b\| \wedge \|b \leq c\| \leq \|a \leq c\|. \quad (15.1)$$

Up to symmetry between P and Q , it is sufficient to consider the three following cases.

Case 1'. $a, b, c \in P$. Then (15.1) follows from the fact that P is a D -valued poset.

Case 2'. $a, b \in P$, and $c \in Q$. We compute:

$$\begin{aligned} \|a \leq b\| \wedge \|b \leq c\| &= \bigvee_{x \in K} \|a \leq b\|_P \wedge \|b \leq x\|_P \wedge \|x \leq c\|_Q \\ &\leq \bigvee_{x \in K} \|a \leq x\|_P \wedge \|x \leq c\|_Q \\ &= \|a \leq c\|. \end{aligned}$$

Case 3'. $a \in P, b, c \in Q$. This case is similar to Case 2'.

Case 4'. $a \in P, b \in Q, c \in P$. For $u, v \in K$,

$$\begin{aligned} \|a \leq u\|_P \wedge \|u \leq b\|_Q \wedge \|b \leq v\|_Q \wedge \|v \leq c\|_P \\ \leq \|a \leq u\|_P \wedge \|u \leq v\|_Q \wedge \|v \leq c\|_P \\ = \|a \leq u\|_P \wedge \|u \leq v\|_P \wedge \|v \leq c\|_P \quad (\text{because } u, v \in K) \\ \leq \|a \leq c\|_P \\ = \|a \leq c\|. \end{aligned}$$

It follows that

$$\begin{aligned}\|a \leq b\| \wedge \|b \leq c\| &= \bigvee_{u, v \in K} \|a \leq u\|_P \wedge \|u \leq b\|_Q \wedge \|b \leq v\|_Q \wedge \|v \leq c\|_P \\ &\leq \|a \leq c\|.\end{aligned}$$

This completes the proof that R is a D -valued poset. Furthermore, it is obvious that $x \leq y$ implies that $\|x \leq y\| = 1$, for all $x, y \in R$.

We now verify items (iv) and (v) of the definition of a D -comeasured partial lattice (see Definition 12.1). Let us verify (iv). So, let $a \in R$ and $X \in [R]_*^{<\omega}$ such that $a = \bigvee X$. We verify that

$$\|a \leq b\| = \bigwedge_{x \in X} \|x \leq b\|, \quad \text{for all } b \in R. \quad (15.2)$$

Without loss of generality, $\{a\} \cup X \subseteq P$ and $a = \bigvee X$ in P . If $b \in P$, then all the Boolean values involved in (15.2) are computed in P , so (15.2) follows from the fact that P is a D -comeasured partial lattice.

So, suppose that $b \in Q$. For $x \in X$, $x \leq a$, thus $\|x \leq a\| = 1$, so $\|a \leq b\| = \|x \leq a\| \wedge \|a \leq b\| \leq \|x \leq b\|$, thus

$$\|a \leq b\| \leq \bigwedge_{x \in X} \|x \leq b\|.$$

To prove the converse inequality, we observe that, since D is distributive and both X and K are nonempty and finite, the following equalities

$$\begin{aligned}\bigwedge_{x \in X} \|x \leq b\| &= \bigwedge_{x \in X} \bigvee_{y \in K} \|x \leq y\|_P \wedge \|y \leq b\|_Q \\ &= \bigvee_{v: X \rightarrow K} \bigwedge_{x \in X} \|x \leq v(x)\|_P \wedge \|v(x) \leq b\|_Q\end{aligned}$$

hold. Thus, to complete the proof of (15.2), it suffices to prove that for any map $v: X \rightarrow K$, the inequality

$$\bigwedge_{x \in X} \|x \leq v(x)\|_P \wedge \|v(x) \leq b\|_Q \leq \|a \leq b\| \quad (15.3)$$

holds. Since K is a lattice, $c = \bigvee_{x \in X} v(x)$ is defined in K . The inequality $\|x \leq v(x)\|_P \leq \|x \leq c\|_P$ holds for any $x \in X$ (because $v(x) \leq c$), and, since $c = \bigvee_{x \in X} v(x)$ in Q and Q is a D -comeasured partial lattice,

$$\bigwedge_{x \in X} \|v(x) \leq b\|_Q = \|c \leq b\|_Q.$$

It follows that

$$\begin{aligned}
\bigwedge_{x \in X} \|x \leq v(x)\|_P \wedge \|v(x) \leq b\|_Q &= \bigwedge_{x \in X} \|x \leq v(x)\|_P \wedge \bigwedge_{x \in X} \|v(x) \leq b\|_Q \\
&\leq \bigwedge_{x \in X} \|x \leq c\|_P \wedge \|c \leq b\|_Q \\
&= \|a \leq c\|_P \wedge \|c \leq b\|_Q
\end{aligned}$$

(because $c = \bigvee_{x \in X} v(x)$ in P and P is a D -comeasured partial lattice)

$$\leq \|a \leq b\|.$$

This completes the proof of (15.2). Therefore, R satisfies (iv) of Definition 12.1. The proof of (v) of Definition 12.1 is dual.

So, R is a D -comeasured partial lattice. The fact that both f' and g' are isometries is trivial.

If S is a D -comeasured partial lattice and $\tilde{f}: P \rightarrow S$ and $\tilde{g}: Q \rightarrow S$ are uniform maps such that $\tilde{f} \circ f = \tilde{g} \circ g$, then there exists a unique map $h: R \rightarrow S$ such that $h \circ f' = \tilde{f}$ and $h \circ g' = \tilde{g}$, namely, h is defined by the rule

$$h(x) = \begin{cases} \tilde{f}(x), & \text{for any } x \in P, \\ \tilde{g}(x), & \text{for any } x \in Q. \end{cases}$$

Since R is the pushout of P and Q above K in the category of partial lattices (see Proposition 3.4), h is a homomorphism of partial lattices. It remains to prove that $\|x \leq y\|_R \leq \|h(x) \leq h(y)\|_S$, for all $x, y \in R$. If $x, y \in P$, then

$$\|h(x) \leq h(y)\|_S = \|\tilde{f}(x) \leq \tilde{f}(y)\|_S \geq \|x \leq y\|_P = \|x \leq y\|_R.$$

A similar proof applies to the case where $x, y \in Q$. If $x \in P$ and $y \in Q$, we compute:

$$\begin{aligned}
\|h(x) \leq h(y)\|_S &= \|\tilde{f}(x) \leq \tilde{g}(y)\|_S \\
&\geq \bigvee_{z \in K} \|\tilde{f}(x) \leq \tilde{f}(z)\|_S \wedge \|\tilde{g}(z) \leq \tilde{g}(y)\|_S \quad (\text{because } \tilde{f}|_K = \tilde{g}|_K) \\
&\geq \bigvee_{z \in K} \|x \leq z\|_P \wedge \|z \leq y\|_Q \quad (\text{because } \tilde{f} \text{ and } \tilde{g} \text{ are uniform}) \\
&= \|x \leq y\|_R.
\end{aligned}$$

The proof is similar in case $x \in Q$ and $y \in P$. Therefore, h is uniform. So, R is the pushout of P and Q above K in the category of D -comeasured partial lattices. \square

Of course, in accordance with Notation 3.5, we shall also write $R = P \sqcup_K Q$ in the context of Proposition 15.4.

In Lemmas 15.5 and 15.6, let $\mathcal{D} = \langle K, P, Q, f, g \rangle$ be a standard V-formation of D -comeasured partial lattices (so that K is a finite lattice). We put $R = P \sqcup_K Q$, endowed with its structure of D -comeasured partial lattice described in Proposition 15.4.

Lemma 15.5. *The equality*

$$\|x = y\| = \bigvee_{z \in K} \|x = z\| \wedge \|z = y\|$$

holds, for all $x, y \in R$ such that either $x \in P$ and $y \in Q$, or $x \in Q$ and $y \in P$.

Proof. We assume, for example, that $x \in P$ and $y \in Q$. We compute:

$$\begin{aligned} \|x = y\| &= \|x \leq y\| \wedge \|y \leq x\| \\ &= \bigvee_{u, v \in K} \|x \leq u\| \wedge \|u \leq y\| \wedge \|y \leq v\| \wedge \|v \leq x\|. \end{aligned}$$

For $u, v \in K$, $\|x \leq u\| \wedge \|v \leq x\| \leq \|v \leq u\|$, while $\|u \leq y\| \wedge \|y \leq v\| \leq \|u \leq v\|$. Therefore,

$$\begin{aligned} \|x = y\| &= \bigvee_{u, v \in K} \|x \leq u\| \wedge \|u \leq y\| \wedge \|y \leq v\| \wedge \|v \leq x\| \wedge \|u = v\| \\ &= \bigvee_{u, v \in K} \|x \leq u\| \wedge \|u \leq y\| \wedge \|y \leq u\| \wedge \|u \leq x\| \wedge \|u = v\| \\ &= \bigvee_{u \in K} \|x = u\| \wedge \|u = y\|, \end{aligned}$$

which concludes the proof. \square

Lemma 15.6.

- (i) *For any $a \in P$ and any $Y \in [Q]_*^{<\omega}$, the inequality $\|a \in Y\| \leq \|a \in K\|$ holds.*
- (ii) *For any $X \in [P]_*^{<\omega}$ and any $Y \in [Q]_*^{<\omega}$, the inequality $\|X \subseteq Y\| \leq \|X \subseteq K\|$ holds.*

Proof. (i) For all $y \in Y$, we compute, using Lemma 15.5:

$$\|a = y\| = \bigvee_{z \in K} \|a = z\| \wedge \|z = y\| \leq \|a \in K\|,$$

since $\|a = z\| \leq \|a \in K\|$. This holds for all $y \in Y$, so the conclusion follows.

(ii) is a trivial consequence of (i). \square

We are now able to prove the following fundamental result.

Proposition 15.7. *Let $\mathcal{D} = \langle K, P, Q, f, g \rangle$ be a standard V-formation of D-comeasured partial lattices. Put $R = P \sqcup_K Q$. If P and Q are finitely covering, then R is finitely covering.*

Proof. We prove, for example, that the domain of the join in R is finitely covering. So let $Z \in [R]_*^{\leq \omega}$, we prove that Z has a $\text{dom } \bigvee_R$ -cover.

Write $Z = X \cup Y$, where $X \in [P]^{< \omega}$ and $Y \in [Q]^{< \omega}$. By Lemma 12.3, applied within P , there exists a finite subset $\{X_i \mid i < m\}$ (with $m > 0$) of $\text{dom } \bigvee_P$ such that

$$\|U \subseteq X \cup K\| \leq \bigvee_{i < m} \|U = X_i\|, \quad \text{for all } U \in \text{dom } \bigvee_P.$$

Therefore, for any $U \in \text{dom } \bigvee_P$,

$$\begin{aligned} \|Z = U\| &\leq \|U \subseteq X \cup Y\| \wedge \|Y \subseteq U\| \\ &\leq \|U \subseteq X \cup Y\| \wedge \|Y \subseteq K\| \quad (\text{by Lemma 15.6(ii)}) \\ &= \|U \subseteq X \cup Y\| \wedge \|X \cup Y \subseteq X \cup K\| \\ &\leq \|U \subseteq X \cup K\| \\ &\leq \bigvee_{i < m} \|U = X_i\|. \end{aligned} \tag{15.4}$$

Similarly, there exists a finite subset $\{X_i \mid m \leq i < m + n\}$ (with $n > 0$) of $\text{dom } \bigvee_Q$ such that

$$\|Z = U\| \leq \bigvee_{j < n} \|U = X_{m+j}\|, \quad \text{for all } U \in \text{dom } \bigvee_Q. \tag{15.5}$$

Therefore, by (15.4) and (15.5) and since $\text{dom } \bigvee_R = \text{dom } \bigvee_P \cup \text{dom } \bigvee_Q$,

$$\|Z = U\| \leq \bigvee_{i < m+n} \|U = X_i\|, \quad \text{for all } U \in \text{dom } \bigvee_R.$$

So, $\{X_i \mid i < m + n\}$ is a $\text{dom } \bigvee_R$ -cover of Z . The proof for the meet is dual. \square

In the context of Proposition 15.7, since R is a finitely covering D -comeasured partial lattice, it can, by Proposition 12.5, be canonically extended into a D -valued partial lattice, namely, \tilde{R} . Hence, for every prime filter G of D , K/G is a finite lattice, and P/G , Q/G , and R/G are partial lattices, see Proposition 5.6. Furthermore, by Lemma 14.3, the canonical maps $K/G \hookrightarrow P/G$, $K/G \hookrightarrow Q/G$, $P/G \hookrightarrow R/G$, and $Q/G \hookrightarrow R/G$ are *embeddings* of partial lattices. The question whether they form a pushout (in the category of partial lattices) is answered naturally as follows.

Proposition 15.8. *Let $\langle K, P, Q \rangle$ be a standard V -formation of D -comeasured partial lattices, with P and Q finitely covering. Then $R/G = P/G \sqcup_{K/G} Q/G$, for any prime filter G of D .*

Proof. From $R = P \cup Q$ follows trivially that $R/G = P/G \cup Q/G$.

Let $\mathbf{a} \in P/G$ and $\mathbf{b} \in Q/G$ such that $\mathbf{a} \leq \mathbf{b}$. Pick $a \in \mathbf{a}$ and $b \in \mathbf{b}$, then $\|a \leq b\| \in G$, thus, by the definition of the Boolean values in R (see Proposition 15.4), there exists $c \in K$ such that $\|a \leq c\| \wedge \|c \leq b\| \in G$. Hence, for $\mathbf{c} = c/G$, we obtain that $\mathbf{a} \leq \mathbf{c} \leq \mathbf{b}$. A similar statement, with P and Q exchanged, holds. This implies that $K/G = P/G \cap Q/G$ and that the ordering of R/G is the same as the ordering of $P/G \sqcup_{K/G} Q/G$, see Proposition 3.4. Finally, the fact that R/G and $P/G \sqcup_{K/G} Q/G$ have the same join and meet operations follows easily from Lemma 14.1. \square

16. (Id \cap)- and (Fil \cap)-samples for $P \sqcup_K Q$

In this section, we shall fix a distributive lattice D with unit and a standard V-formation $\langle K, P, Q \rangle$ of D -comeasured partial lattices, with P and Q finitely covering. We put $R = P \sqcup_K Q$. By Propositions 15.4 and 15.7, R is a finitely covering D -comeasured partial lattice.

This section will be devoted to the proof of the following result.

Proposition 16.1. *Suppose that P and Q have (Id \cap) (respectively (Fil \cap)). Then R has (Id \cap) (respectively (Fil \cap)).*

Proof. We provide a proof for (Id \cap); the proof for (Fil \cap) is dual.

Let $a, b \in R$, we shall find an (Id \cap)-sample of $\{a, b\}$. Up to symmetry between P and Q , there are two cases to consider.

Case 1. $a, b \in P$.

Let U be an (Id \cap)-sample of $\{a, b\}$ in P , we prove that U is also an (Id \cap)-sample of $\{a, b\}$ in R . This amounts to proving the inequality

$$\|x \leq a\| \wedge \|x \leq b\| \leq \bigvee_{u \in U} \|x \leq u\| \wedge \|u \leq a\| \wedge \|u \leq b\|, \quad \text{for all } x \in R \quad (16.1)$$

(the converse inequality of (16.1) is trivial). First, for $x \in P$, all the Boolean values involved in (16.1) are Boolean values in P , so, since U is an (Id \cap)-sample of $\{a, b\}$ in P , (16.1) holds.

Now suppose that $x \in Q$. We compute:

$$\begin{aligned} \|x \leq a\| \wedge \|x \leq b\| &= \bigvee_{u, v \in K} \|x \leq u\| \wedge \|u \leq a\| \wedge \|x \leq v\| \wedge \|v \leq b\| \\ &= \bigvee_{w \in K} \|x \leq w\| \wedge \|w \leq a\| \wedge \|w \leq b\| \end{aligned}$$

(by putting $w = u \wedge v$ in K)

$$= \bigvee_{w \in K, u \in U} \|x \leq w\| \wedge \|w \leq u\| \wedge \|u \leq a\| \wedge \|u \leq b\|$$

(because U is an $(\text{Id}\cap)$ -sample of $\{a, b\}$ in P)

$$\leq \bigvee_{u \in U} \|x \leq u\| \wedge \|u \leq a\| \wedge \|u \leq b\|,$$

so (16.1) is established in this case.

Case 2. $a \in P, b \in Q$.

Let U (respectively V) be a common $(\text{Id}\cap)$ -sample in P (respectively in Q) of all pairs of the form $\{a, z\}$ (respectively $\{b, z\}$), for $z \in K$. We put $W = U \cup V$, and we prove that W is an $(\text{Id}\cap)$ -sample of $\{a, b\}$ in R .

For any $x \in P$, we compute:

$$\begin{aligned} \|x \leq a\| \wedge \|x \leq b\| &= \bigvee_{z \in K} \|x \leq a\| \wedge \|x \leq z\| \wedge \|z \leq b\| \\ &= \bigvee_{z \in K, u \in U} \|x \leq u\| \wedge \|u \leq a\| \wedge \|u \leq z\| \wedge \|z \leq b\| \end{aligned}$$

(because U is an $(\text{Id}\cap)$ -sample of all pairs $\{a, z\}$ for $z \in K$)

$$\leq \bigvee_{u \in U} \|x \leq u\| \wedge \|u \leq a\| \wedge \|u \leq b\|.$$

Similarly, we can obtain that

$$\|x \leq a\| \wedge \|x \leq b\| \leq \bigvee_{v \in V} \|x \leq v\| \wedge \|v \leq a\| \wedge \|v \leq b\|,$$

for any $x \in Q$. Hence,

$$\|x \leq a\| \wedge \|x \leq b\| \leq \bigvee_{w \in W} \|x \leq w\| \wedge \|w \leq a\| \wedge \|w \leq b\|,$$

for any $x \in R$. \square

17. $(\text{Id}\vee)$ - and $(\text{Fil}\vee)$ -samples in $P \amalg_K Q$

In this section, we shall fix, as in Section 16, a distributive lattice D with unit and a standard V-formation $\langle K, P, Q \rangle$ of D -comeasured partial lattices, with P and Q finitely covering. We put $R = P \amalg_K Q$. By Propositions 15.4 and 15.7, R is a finitely covering D -comeasured partial lattice.

This section will be devoted to the proof of the following result.

Proposition 17.1. *Suppose that P and Q have $(\text{Id}\vee)$ (respectively $(\text{Fil}\vee)$). Then R has $(\text{Id}\vee)$ (respectively $(\text{Fil}\vee)$).*

Proof. We provide a proof for $(\text{Id}\vee)$; the proof for $(\text{Fil}\vee)$ is dual.

Let $Z \in [R]_*^{<\omega}$. We put $X = Z \cap P$ and $Y = Z \cap Q$. Observe that $Z = X \cup Y$.

Let X^* be a common $(\text{Id}\vee)$ -sample of all subsets of $X \cup K$ in P . Symmetrically, let Y^* be a common $(\text{Id}\vee)$ -sample of all subsets of $Y \cup K$ in Q . Let m be a common index for both samples, see Definition 7.6.

We denote by h the height of K , and we put $k = (h + 2)m + h + 1$. We shall prove the following assertion:

$$Z^* = X^* \cup Y^* \text{ is an } (\text{Id}\vee)\text{-sample of } Z \text{ in } R, \text{ with index } k. \quad (17.1)$$

Let G be a prime filter of D . We put $\bar{T} = T/G$, for every subset T of R . Recall that $\bar{R} = \bar{P} \sqcup_{\bar{K}} \bar{Q}$, see Proposition 15.8. We define $I_0 \subseteq \bar{P}$ and $J_0 \subseteq \bar{Q}$ as follows:

$$I_0 = \text{Id}_m^{\bar{P}}(\bar{X}, \bar{X}^*); \quad J_0 = \text{Id}_m^{\bar{Q}}(\bar{Y}, \bar{Y}^*).$$

Of course, the superscript \bar{P} (or \bar{Q}) on the math operator Id indicates in which partial lattice the $\text{Id}_n(U, V)$ function (see Definition 2.9) is computed.

Since m is an index for the $(\text{Id}\vee)$ -sample X^* of X , it follows from Lemma 7.5 that

$$\text{Id}_m^{\bar{P}}(\bar{X}, \bar{X}^*) = \text{Id}_{m+1}^{\bar{P}}(\bar{X}, \bar{T}),$$

for every $T \in [P]_*^{<\omega}$ such that $X^* \subseteq T$. In particular, I_0 is an ideal of \bar{P} (it is empty if $X = \emptyset$). Similarly, J_0 is an ideal of \bar{Q} .

Claim 1. Assume that $(I_0 \cup J_0) \cap \bar{K} = \emptyset$. Then $I_0 \cup J_0$ is an ideal of \bar{R} . Furthermore, $I_0 \cup J_0 = \text{Id}_n^{\bar{R}}(\bar{Z}, \bar{T})$ holds for all $n \geq m$ and all $T \supseteq Z^*$ in $[R]_*^{<\omega}$.

Proof of Claim 1. Since $\bigvee \bar{R} = \bigvee \bar{P} \cup \bigvee \bar{Q}$ and since I_0 (respectively J_0) is an ideal of \bar{P} (respectively \bar{Q}), $I_0 \cup J_0$ is closed under $\bigvee \bar{R}$. By the assumption that $(I_0 \cup J_0) \cap \bar{K} = \emptyset$, no pair of $I_0 \times \bar{Q}$ and $\bar{P} \times J_0$ is comparable, so, since I_0 (respectively J_0) is a lower subset of \bar{P} (respectively \bar{Q}), $I_0 \cup J_0$ is a lower subset of \bar{R} . Every ideal of \bar{R} that contains \bar{Z} contains $I_0 \cup J_0$, thus, since $\bar{Z} \subseteq I_0 \cup J_0$, $I_0 \cup J_0$ is the ideal of \bar{R} generated by \bar{Z} . The second part of the statement of Claim 1 follows immediately. \square

Now we assume that $(I_0 \cup J_0) \cap \bar{K}$ is nonempty. Since it is a nonempty subset of the finite lattice \bar{K} , it admits a supremum, that we denote by c_0 . We observe that both I_0 and J_0 are contained in $\text{Id}_m^{\bar{R}}(\bar{Z}, \bar{Z}^*)$, thus $c_0 \in \text{Id}_{m+1}^{\bar{R}}(\bar{Z}, \bar{Z}^*)$. We extend this construction by defining inductively I_n , J_n , and c_n , for any $n < \omega$, by

$$I_{n+1} = \text{Id}_m^{\bar{P}}(\bar{X} \cup \{c_n\}, \bar{X}^*), \quad J_{n+1} = \text{Id}_m^{\bar{Q}}(\bar{Y} \cup \{c_n\}, \bar{Y}^*), \quad c_n = \bigvee ((I_n \cup J_n) \cap K),$$

for all $n < \omega$.

Since X^* is a common $(\text{Id}\vee)$ -sample of all subsets of $X \cup K$ in P , with index m , I_{n+1} is an ideal of P , for all $n < \omega$. Similarly, J_{n+1} is an ideal of Q . Furthermore, $c_n \in I_{n+1} \cap J_{n+1}$, thus $c_n \leq c_{n+1}$. Therefore, $I_n \subseteq I_{n+1}$ and $J_n \subseteq J_{n+1}$ for all n .

An easy inductive generalization of the argument above showing that $c_0 \in \text{Id}_{m+1}^{\bar{R}}(\bar{Z}, \bar{Z}^*)$ leads to the following claim.

Claim 2. $I_n \cup J_n \subseteq \text{Id}_{(n+1)m+n}^{\bar{R}}(\bar{Z}, \bar{Z}^*)$, for all $n < \omega$.

For $n < \omega$, if $c_0 < c_1 < \dots < c_n$, then $n < \text{ht}(\bar{K}) \leq \text{ht } K = h$. If $c_n = c_{n+1}$, then $c_n = c_l$ for all $l \geq n$. It follows from this that $c_h = c_{h+1}$, thus $I_{h+1} = I_{h+2}$ and $J_{h+1} = J_{h+2}$. We put $c = c_h$, $I = I_{h+1}$, and $J = J_{h+1}$.

Claim 3. $I \cup J$ is an ideal of \bar{R} .

Proof of Claim 3. Since I is an ideal of \bar{P} and J is an ideal of \bar{Q} , $I \cup J$ is closed under $\bigvee_{\bar{R}}$. Now we prove that $I \cup J$ is a lower subset of \bar{R} . Since $c_0 \in I_1 \cap J_1 \subseteq I \cap J$, both $I \cap \bar{K}$ and $J \cap \bar{K}$ are nonempty, hence there are elements a and b of \bar{K} defined by

$$a = \bigvee (I \cap \bar{K}) \quad \text{and} \quad b = \bigvee (J \cap \bar{K}).$$

Since $c = c_h \in I \cap J \cap \bar{K}$, $c \leq a$ and $c \leq b$. On the other hand,

$$c = \bigvee ((I \cup J) \cap \bar{K}) = a \vee b,$$

whence $c = a = b$. So we have established that

$$c = \bigvee (I \cap \bar{K}) = \bigvee (J \cap \bar{K}). \quad (17.2)$$

Now let $x \in \bar{R}$, $y \in I \cup J$ such that $x \leq y$, we prove that $x \in I \cup J$. By symmetry, we may assume that $y \in I$. If $x \in \bar{P}$, then, since I is a lower subset of \bar{P} , $x \in I$ and we are done. If $x \in \bar{Q}$, then there exists $z \in \bar{K}$ such that $x \leq z \leq y$. Since $z \in \bar{K}$ and $z \leq y \in I$, $z \leq c$ by (17.2), so $x \leq c$. But $c \in J$ and J is a lower subset of \bar{Q} , thus $x \in J$, and we are done again.

So $I \cup J$ is a lower subset of \bar{R} , hence an ideal of \bar{R} . \square

By Claims 2 and 3, $I \cup J = \text{Id}_{(h+2)m+h+1}^{\bar{R}}(\bar{Z}, \bar{T})$, for all finite T containing Z^* , is the ideal of \bar{R} generated by \bar{Z} . In particular,

$$\text{Id}_{(h+2)m+h+1}^{\bar{R}}(\bar{Z}, \bar{Z}^*) = \text{Id}_{(h+2)m+h+2}^{\bar{R}}(\bar{Z}, \bar{T}), \quad (17.3)$$

for all finite T containing Z^* . This also holds in the context of Claim 1, since one can, in that case, replace $(h+2)m+h+1$ by m . Therefore, (17.3) holds for every prime filter G of D . By Lemma 7.9, this proves (17.1). \square

Part 4. Congruence amalgamation with distributive target

18. Proof of Theorem B

We first observe the following obvious restatement of Theorem B in terms of D -measured partial lattices.

Theorem B. *Let D be a distributive lattice with zero. Let $\langle K, \lambda \rangle$, $\langle P, \mu \rangle$, and $\langle Q, \nu \rangle$ be D -measured partial lattices, with K a finite lattice and each of P and Q either a finite partial lattice or a lattice. Let $f: \langle K, \lambda \rangle \rightarrow \langle P, \mu \rangle$ and $g: \langle K, \lambda \rangle \rightarrow \langle Q, \nu \rangle$ be homomorphisms.*

Then there exist a D -measured lattice $\langle L, \varphi \rangle$ and homomorphisms $\bar{f}: \langle P, \mu \rangle \rightarrow \langle L, \varphi \rangle$ and $\bar{g}: \langle Q, \nu \rangle \rightarrow \langle L, \varphi \rangle$ such that $\bar{f} \circ f = \bar{g} \circ g$. Furthermore, the construction can be done in such a way that the following additional properties hold:

- (i) L is generated, as a lattice, by $\bar{f}[P] \cup \bar{g}[Q]$.
- (ii) The map φ isolates 0.

By Proposition 13.1, if D is a distributive lattice with zero, then the notions of D -measured partial lattice (see Definition 13.2) and of D^d -comeasured partial lattice (see Definition 12.1) are, essentially, equivalent. It is, in fact, easy to see that this is a category equivalence. The corresponding notion of *morphism of D -measured partial lattice* is given by the following very easy result.

Lemma 18.1. *Let D be a distributive lattice with zero, let $\langle P, \mu \rangle$ and $\langle Q, \nu \rangle$ be D -measured partial lattices, let $f: P \hookrightarrow Q$ be an embedding of partial lattices. Then the following are equivalent:*

- (i) *The equality $\nu \circ \text{Con}_c f = \mu$ holds.*
- (ii) *If $\langle P, \mu \rangle$ and $\langle Q, \nu \rangle$ are viewed as D^d -comeasured partial lattices, then f is an isometry (see Definition 14.2).*

Proof. Endow each of the structures P and Q with its map $\|-\|$, with target D^d . An explicit definition of $\|-\|_P$ and $\|-\|_Q$ is the following:

$$\begin{aligned} \|x \leq y\|_P &= \mu(\Theta_P^+(x, y)), \quad \text{for all } x, y \in P; \\ \|x \leq y\|_Q &= \nu(\Theta_Q^+(x, y)), \quad \text{for all } x, y \in Q. \end{aligned}$$

Hence, for $x, y \in P$,

$$\|f(x) \leq f(y)\|_Q = \nu(\Theta_Q^+(f(x), f(y))) = \nu \circ (\text{Con}_c f)(\Theta_P^+(x, y)). \quad (18.1)$$

But the principal congruences $\Theta_P^+(x, y)$, for $x, y \in P$, generate the $\{\vee, 0\}$ -semilattice $\text{Con}_c P$. Hence, $\nu \circ \text{Con}_c f = \mu$ iff both maps $\nu \circ \text{Con}_c f$ and μ agree on all principal congruences of P , that is, by (18.1), $\|f(x) \leq f(y)\|_Q = \|x \leq y\|_P$, for all $x, y \in P$. \square

Definition 18.2. Let D be a distributive lattice with zero, let $\langle P, \mu \rangle$ and $\langle Q, \nu \rangle$ be D -measured partial lattices. A *homomorphism* from $\langle P, \mu \rangle$ to $\langle Q, \nu \rangle$ is a homomorphism $f: P \rightarrow Q$ of partial lattices such that $\nu \circ \text{Con}_c f = \mu$. If, in addition, f is an embedding of partial lattices, we say that f is an *embedding* of D -measured partial lattices.

Definition 18.3. Let D be a distributive lattice with zero, let $\langle P, \mu \rangle$ be a D -measured partial lattice. The *kernel* of $\langle P, \mu \rangle$ is the congruence θ of P defined by the rule

$$x \leq_\theta y \quad \text{iff} \quad \mu \Theta_P^+(x, y) = 0, \quad \text{for all } x, y \in P.$$

The *kernel projection* of $\langle P, \mu \rangle$ is the canonical projection from P onto P/θ .

In other words, the kernel of $\langle P, \mu \rangle$ is the largest congruence θ of P such that $\mu(\xi) = 0$ for all $\xi \leq \theta$ in $\text{Con}_c P$. In particular, $\langle P, \mu \rangle$ is proper (see Definition 13.2) iff its kernel projection is trivial.

The following lemma states that the kernel projection of $\langle P, \mu \rangle$ is the universal projection of $\langle P, \mu \rangle$ onto a proper D -measured partial lattice.

Lemma 18.4. Let D be a distributive lattice with zero, let $\langle P, \mu \rangle$ be a D -measured partial lattice. We denote by $p: P \rightarrow P'$ the kernel projection of P . Then there exists a unique $\{\vee, 0\}$ -homomorphism $\mu': \text{Con}_c P' \rightarrow D$ such that $\mu' \circ \text{Con}_c p = \mu$. Furthermore, the following assertions hold:

- (i) $\langle P', \mu' \rangle$ is a proper D -measured partial lattice.
- (ii) For every proper D -measured partial lattice $\langle Q, \nu \rangle$ and every homomorphism $f: \langle P, \mu \rangle \rightarrow \langle Q, \nu \rangle$, there exists a unique homomorphism $f': \langle P', \mu' \rangle \rightarrow \langle Q, \nu \rangle$ such that $f' \circ p = f$, and f' is an embedding of D -measured partial lattices (see Definition 18.2).

Proof. (i) The compact congruences of $P' = P/\theta$ are exactly the congruences of the form $\alpha \vee \theta/\theta$, where α is a compact congruence of P . By the definition of θ , $\alpha \vee \theta \leq \beta \vee \theta$ implies that $\mu(\alpha) \leq \mu(\beta)$, for all $\alpha, \beta \in \text{Con}_c P$. Hence we can define a $\{\vee, 0\}$ -homomorphism $\mu': \text{Con}_c P' \rightarrow D$ by the rule

$$\mu'(\alpha \vee \theta/\theta) = \mu(\alpha), \quad \text{for all } \alpha \in \text{Con}_c P.$$

Observe that $\mu' \circ \text{Con}_c p = \mu$. The uniqueness assertion about μ' follows from the surjectivity of the map $\text{Con}_c p$.

(ii) For all $x, y \in P$, $x/\theta \leq y/\theta$ iff $\mu \Theta_P^+(x, y) = 0$, that is, $\nu \Theta_Q^+(f(x), f(y)) = 0$, or, since $\langle Q, \nu \rangle$ is proper, $f(x) \leq f(y)$. This makes it possible to define an embedding $f': P' \hookrightarrow Q$ of partial lattices by the rule $f'(x/\theta) = f(x)$, for all $x \in P$, and $f' \circ p = f$.

Observe that $v \circ \text{Conc } f' = \mu'$. The uniqueness assertion about f' follows from the surjectivity of the map p . \square

Proposition 18.5. *Let D be a distributive lattice with zero. Let $\langle K, \lambda \rangle$, $\langle P, \mu \rangle$, and $\langle Q, v \rangle$ be D -measured partial lattices, with K a finite lattice and P, Q balanced. Let $f: \langle K, \lambda \rangle \rightarrow \langle P, \mu \rangle$ and $g: \langle K, \lambda \rangle \rightarrow \langle Q, v \rangle$ be homomorphisms.*

Then there exists a proper D -measured lattice $\langle L, \varphi \rangle$, together with homomorphisms $\bar{f}: \langle P, \mu \rangle \rightarrow \langle L, \varphi \rangle$ and $\bar{g}: \langle Q, v \rangle \rightarrow \langle L, \varphi \rangle$, such that $\bar{f} \circ f = \bar{g} \circ g$ and L is generated, as a lattice, by $\bar{f}[P] \cup \bar{g}[Q]$.

Proof. We first consider the case where f and g are embeddings and λ, μ , and v isolate 0. We view $\langle K, \lambda \rangle$, $\langle P, \mu \rangle$, and $\langle Q, v \rangle$ as D^d -comeasured partial lattices. By Lemma 15.3, we can assume without loss of generality that $\langle K, P, Q, f, g \rangle$ is a standard V-formation. Now we put $R = P \sqcup_K Q$, as defined in Proposition 15.4, with the corresponding embeddings f' and g' . Let $\langle R, \varrho \rangle$ be the corresponding D -measured partial lattice. From the fact that both μ and v isolate 0 and the description of R (Proposition 15.4) follows that ϱ isolates 0. By Propositions 15.7, 16.1, and 17.1, R is balanced. By Theorem A, there exists a $\{\vee, 0\}$ -homomorphism $\varphi': \text{Conc } \mathbf{F}_L(R) \rightarrow D$ such that $\varphi' \circ \text{Conc } j_R = \varrho$. We denote by $p: \mathbf{F}_L(R) \rightarrow L$ the kernel projection of $\langle \mathbf{F}_L(R), \varphi' \rangle$, see Definition 18.3, and we put $j = p \circ j_R$. By Lemma 18.4, there exists a unique $\{\vee, 0\}$ -homomorphism $\varphi: \text{Conc } L \rightarrow D$ such that $\varphi \circ \text{Conc } p = \varphi'$, and $\langle L, \varphi \rangle$ is proper. Furthermore,

$$\varphi \circ \text{Conc } j = \varphi \circ \text{Conc } p \circ \text{Conc } j_R = \varphi' \circ \text{Conc } j_R = \varrho.$$

We put $\bar{f} = j \circ f'$ and $\bar{g} = j \circ g'$. From $f' \circ f = g' \circ g$ follows that $\bar{f} \circ f = \bar{g} \circ g$. Since $j_R[R]$ generates $\mathbf{F}_L(R)$, $\bar{f}[P] \cup \bar{g}[Q]$ generates L . Since j is a homomorphism from $\langle R, \varrho \rangle$ to $\langle L, \varphi \rangle$ and since both ϱ and φ isolate 0, j is an embedding, thus \bar{f} and \bar{g} are embeddings.

Now we consider the general case. Let $h: \langle K, \lambda \rangle \rightarrow \langle K', \lambda' \rangle$, $p: \langle P, \mu \rangle \rightarrow \langle P', \mu' \rangle$, $q: \langle Q, v \rangle \rightarrow \langle Q', v' \rangle$ be the kernel projections. By Lemma 18.4, there are embeddings $f': \langle K', \lambda' \rangle \hookrightarrow \langle P', \mu' \rangle$ and $g': \langle K', \lambda' \rangle \hookrightarrow \langle Q', v' \rangle$ such that $f' \circ h = p \circ f$ and $g' \circ h = q \circ g$. By the result of the previous paragraph, there exist a proper D -measured lattice $\langle L, \varphi \rangle$ and embeddings $f'': \langle P', \mu' \rangle \hookrightarrow \langle L, \varphi \rangle$ and $g'': \langle Q', v' \rangle \hookrightarrow \langle L, \varphi \rangle$ such that $f'' \circ f' = g'' \circ g'$ and L is generated by $f''[P'] \cup g''[Q']$. We put $\bar{f} = f'' \circ p$ and $\bar{g} = g'' \circ q$. \square

Remark 18.6. In the context of Proposition 18.5, we shall later make use of the following simple fact: *If $\langle Q, v \rangle$ is proper, then \bar{g} is an embedding.*

Indeed, for all $x, y \in Q$,

$$\begin{aligned} \bar{g}(x) \leq \bar{g}(y) &\Rightarrow \varphi \Theta_L^+(\bar{g}(x), \bar{g}(y)) = 0 \quad (\text{because } \varphi \text{ isolates } 0) \\ &\Leftrightarrow (\varphi \circ \text{Conc } \bar{g})(\Theta_Q^+(x, y)) = 0 \\ &\Leftrightarrow v \Theta_Q^+(x, y) = 0 \\ &\Leftrightarrow x \leq y \quad (\text{because } \langle Q, v \rangle \text{ is proper}), \end{aligned}$$

which proves our assertion.

Proof of Theorem B. By Proposition 12.7, every D -measured partial lattice $\langle R, \varrho \rangle$ such that either R is finite with nonempty join and meet operations or R is a lattice is balanced. Theorem B follows immediately as a particular case of Proposition 18.5. \square

19. Saturation properties of D -measured partial lattices

We start with a definition.

Definition 19.1. Let D be a distributive lattice with zero, let $\langle P, \mu \rangle$ and $\langle L, \varphi \rangle$ be D -measured partial lattices, with L a lattice. We say that an embedding $f: \langle P, \mu \rangle \hookrightarrow \langle L, \varphi \rangle$ is a *lower embedding* (respectively *upper embedding*, *internal embedding*), if the filter (respectively ideal, convex sublattice) of L generated by P equals L .

We refer to Definition 18.2 for the definition of an embedding of D -measured partial lattices.

Definition 19.2. Let D be a distributive lattice with zero. A proper D -measured lattice $\langle L, \varphi \rangle$ is *saturated* (respectively *lower saturated*, *upper saturated*, *internally saturated*), if for every embedding (respectively lower embedding, upper embedding, internal embedding) $e: \langle K, \lambda \rangle \hookrightarrow \langle P, \mu \rangle$ of finite proper D -measured partial lattices, with K a lattice, and every homomorphism $f: \langle K, \lambda \rangle \rightarrow \langle L, \varphi \rangle$, there exists a homomorphism $g: \langle P, \mu \rangle \rightarrow \langle L, \varphi \rangle$ such that $g \circ e = f$.

Proposition 19.3. Let D be a distributive lattice with zero. Every proper balanced D -measured partial lattice $\langle P, \varphi \rangle$ admits an embedding (respectively a lower embedding, an upper embedding, an internal embedding) into a saturated (respectively lower saturated, upper saturated, internally saturated) D -measured lattice $\langle L, \psi \rangle$ such that $|L| = |P| + |D| + \aleph_0$.

Proof. A standard increasing chain argument. We present the proof for saturated, the proofs for lower, upper, or internally saturated are similar. We put $\kappa = |P| + |D| + \aleph_0$. By Theorem A, there exist a D -measured lattice $\langle K, \zeta \rangle$ and an embedding $f: \langle P, \varphi \rangle \hookrightarrow \langle K, \zeta \rangle$. Furthermore, by replacing K by the image of its kernel projection (use Lemma 18.4), we can suppose that $\langle K, \zeta \rangle$ is proper. Hence, without loss of generality, P is a lattice.

Let $e_\xi: \langle K_\xi, \lambda_\xi \rangle \hookrightarrow \langle Q_\xi, \nu_\xi \rangle$, $f_\xi: \langle K_\xi, \lambda_\xi \rangle \rightarrow \langle P, \varphi \rangle$, for $\xi < \kappa$, enumerate, up to isomorphism, all embeddings $e: \langle K, \lambda \rangle \hookrightarrow \langle Q, \nu \rangle$ and homomorphisms $f: \langle K, \lambda \rangle \rightarrow \langle P, \varphi \rangle$ with $\langle K, \lambda \rangle$ and $\langle Q, \nu \rangle$ finite, proper D -measured partial lattices, and with K a lattice. It is easy to construct, by using Proposition 18.5 and Remark 18.6, a transfinite chain $\langle \langle L_\xi, \varphi_\xi \rangle \mid \xi < \kappa \rangle$ of proper D -measured lattices, together with embeddings $f_{\xi, \eta}: \langle L_\xi, \varphi_\xi \rangle \hookrightarrow \langle L_\eta, \varphi_\eta \rangle$, for $\xi < \eta < \kappa$, satisfying the following properties:

- (i) $\langle L_0, \varphi_0 \rangle = \langle P, \varphi \rangle$;
- (ii) $f_{\xi, \zeta} = f_{\eta, \zeta} \circ f_{\xi, \eta}$, for $\xi < \eta < \zeta < \kappa$;

- (iii) for any $\xi < \kappa$, there exists a homomorphism $g_\xi : \langle Q_\xi, v_\xi \rangle \rightarrow \langle L_{\xi+1}, \varphi_{\xi+1} \rangle$ such that $f_{0,\xi+1} \circ f_\xi = g_\xi \circ e_\xi$; that is, the following diagram is commutative:

$$\begin{array}{ccc} \langle Q_\xi, v_\xi \rangle & \xrightarrow{g_\xi} & \langle L_{\xi+1}, \varphi_{\xi+1} \rangle \\ e_\xi \uparrow & & \uparrow f_{\xi,\xi+1} \\ \langle K_\xi, \lambda_\xi \rangle & \xrightarrow{f_{0,\xi} \circ f_\xi} & \langle L_\xi, \varphi_\xi \rangle. \end{array}$$

We denote by $\langle P, \varphi \rangle'$ the direct limit of all $\langle L_\xi, \varphi_\xi \rangle$, with transition maps $f_{\xi,\eta}$, for $\xi < \eta < \kappa$. Let $f_{\langle P, \varphi \rangle} : \langle P, \varphi \rangle \hookrightarrow \langle P, \varphi \rangle'$ be the limiting map associated with the direct system above. Observe that $f_{\langle P, \varphi \rangle}$ is an embedding.

The D -measured lattice $\langle P, \varphi \rangle'$ has the property that for every embedding $e : \langle K, \lambda \rangle \hookrightarrow \langle Q, v \rangle$ of finite proper D -measured partial lattices, with K a lattice, and every homomorphism $f : \langle K, \lambda \rangle \rightarrow \langle P, \varphi \rangle$, there exists a homomorphism $g : \langle Q, v \rangle \rightarrow \langle P, \varphi \rangle'$ such that $g \circ e = f_{\langle P, \varphi \rangle} \circ f$.

To conclude the proof, it suffices to iterate the process ω times: put $\langle P^{(0)}, \varphi^{(0)} \rangle = \langle P, \varphi \rangle$, and, for $n < \omega$, put

$$\langle P^{(n+1)}, \varphi^{(n+1)} \rangle = \langle P^{(n)}, \varphi f_{\langle P^{(n)}, \varphi^{(n)} \rangle} : \langle P^{(n)}, \varphi^{(n)} \rangle \hookrightarrow \langle P^{(n+1)}, \varphi^{(n+1)} \rangle \rangle.$$

The direct limit $\langle L, \varphi \rangle$ of all the $\langle P^{(n)}, \varphi^{(n)} \rangle$, with respect to the transition maps $f_{\langle P^{(n)}, \varphi^{(n)} \rangle}$, satisfies the required conditions. \square

20. Proofs of Theorems C and D

We first recall the statement of Theorem C.

Theorem C. *Let K be a lattice, let D be a distributive lattice with zero, and let $\varphi : \text{Con}_c K \rightarrow D$ be a $\{\vee, 0\}$ -homomorphism. There are a relatively complemented lattice L of cardinality $|K| + |D| + \aleph_0$, a lattice homomorphism $f : K \rightarrow L$, and an isomorphism $\alpha : \text{Con}_c L \rightarrow D$ such that the following assertions hold:*

- (i) $\varphi = \alpha \circ \text{Con}_c f$.
- (ii) The range of f is coinitial (respectively cofinal) in L .
- (iii) If the range of φ is cofinal in D , then the range of f is internal in L .

In this section, we shall fix a distributive lattice D with zero and an internally saturated D -measured lattice $\langle L, \varphi \rangle$.

Lemma 20.1. *The lattice L is relatively complemented.*

Proof. Let $a < b < c$ in L , we prove that there exists $x \in L$ such that $a = b \wedge x$ and $c = b \vee x$.

Fig. 1. Adding a relative complement of b in $[a, c]$.

Put $K = \{a, b, c\}$, the three-element chain, let $f: K \hookrightarrow L$ be the inclusion map. If we put $\lambda = \varphi \circ \text{Con}_c f$, then $\langle K, \lambda \rangle$ is a finite, proper (see Definition 19.2) D -measured lattice and f is an embedding from $\langle K, \lambda \rangle$ into $\langle L, \varphi \rangle$.

Next, we put $P = \{a, b, c, t\}$, the two-atom Boolean lattice, with zero element a , unit element c , and atoms b and t , endowed with the homomorphism $\mu: \text{Con}_c P \rightarrow D$ defined by

$$\mu\Theta_P(a, b) = \mu\Theta_P(t, c) = \varphi\Theta_L(a, b),$$

$$\mu\Theta_P(a, t) = \mu\Theta_P(b, c) = \varphi\Theta_L(b, c).$$

Then $\langle P, \mu \rangle$ is a proper D -measured lattice, and the inclusion map $j: K \hookrightarrow P$ is an embedding from $\langle K, \lambda \rangle$ into $\langle P, \mu \rangle$. The lattices K and P can be visualized on Fig. 1.

By assumption on $\langle L, \varphi \rangle$, there exists a homomorphism $g: \langle P, \mu \rangle \rightarrow \langle L, \varphi \rangle$ such that $g \circ j = f$. Put $x = g(t)$. Then $a = b \wedge x$ and $c = b \vee x$. \square

Definition 20.2. Let $o \leq i$ be elements of a lattice K . We say that the elements a, b of the interval $[o, i]$ are *perspective in* $[o, i]$, if there exists $x \in [o, i]$ such that $x \wedge a = x \wedge b$ and $x \vee a = x \vee b$.

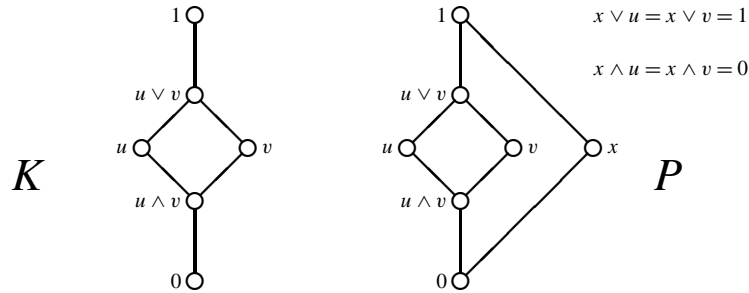
Lemma 20.3. Let $o, i, a, b \in L$ such that $o \leq \{a, b\} \leq i$. Then the following conditions are equivalent:

- (i) a and b are perspective in $[o, i]$.
- (ii) $\varphi\Theta_L(o, a) = \varphi\Theta_L(o, b)$ and $\varphi\Theta_L(a, i) = \varphi\Theta_L(b, i)$.

Proof. (i) \Rightarrow (ii). If a and b are perspective in $[o, i]$, then the intervals $[o, a]$ and $[o, b]$ are projective, hence $\Theta_L(o, a) = \Theta_L(o, b)$. Similarly, $\Theta_L(a, i) = \Theta_L(b, i)$.

(ii) \Rightarrow (i). Let $K = \{0, u \wedge v, u, v, u \vee v, 1\}$ be the lattice diagrammed on Fig. 2, and let $f: K \rightarrow L$ be the unique lattice homomorphism sending 0 to o , 1 to i , u to a , and v to b . We put $\lambda = \varphi \circ \text{Con}_c f$.

If we could find a finite proper D -measured partial lattice $\langle P, \mu \rangle$ and an internal embedding $j: \langle K, \lambda \rangle \hookrightarrow \langle P, \mu \rangle$ such that u and v are perspective in P , then an argument similar to the one used in the proof of Lemma 20.1 would conclude the proof.

Fig. 2. Making u and v perspective in $[0, 1]$.

To this end, we simply put $P = K \cup \{x\}$, for an element x not in K , with the ordering of K extended by the relations $0 < x < 1$, together with the following additional joins and meets:

$$x \vee u = x \vee v = 1; \quad x \wedge u = x \wedge v = 0, \quad (20.1)$$

see Fig. 2. We denote by $j : K \hookrightarrow P$ the canonical embedding. Observe that j is internal.

We claim that the map $\text{Conc } j$ is surjective. Indeed, it is easy to verify that the following equalities hold:

$$\begin{aligned} \Theta_P^+(x, u \vee v) &= \Theta_P(u \vee v, 1); \\ \Theta_P^+(x, u) &= \Theta_P^+(x, v) = \Theta_P^+(x, u \wedge v) = \Theta_P^+(x, 0) = \Theta_P(u, 1) = \Theta_P(v, 1), \end{aligned}$$

thus all the congruences $\Theta_P^+(x, w)$, for $w \in K$, belong to the range of $\text{Conc } j$. A similar statement applies to the congruences $\Theta_P^+(w, x)$, for $w \in K$, which establishes our claim.

We now define congruences ξ, η, α , and β of P by

$$\xi = \Theta_P(0, u \wedge v); \quad \eta = \Theta_P(u \vee v, 1); \quad \alpha = \Theta_P^+(u, v); \quad \beta = \Theta_P^+(v, u).$$

It follows from (20.1) that $\xi \vee \alpha = \xi \vee \beta$ —denote it by $\bar{\xi}$, and that $\eta \vee \alpha = \eta \vee \beta$ —denote it by $\bar{\eta}$. Therefore, by using the surjectivity of $\text{Conc } j$, we obtain that

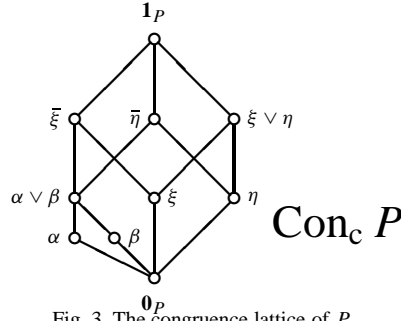
$$\text{Conc } P = \{0_P, \xi, \eta, \alpha, \beta, \alpha \vee \beta, \bar{\xi}, \bar{\eta}, 1_P\}, \quad (20.2)$$

with all the elements of the right-hand side of (20.2) pairwise distinct. The lattice $\text{Conc } P$ is diagrammed on Fig. 3.

Hence $\text{Conc } P$ is the $\{\vee, 0\}$ -semilattice freely generated by $\check{\xi}, \check{\eta}, \check{\alpha}, \check{\beta}$, subject to the relations

$$\check{\xi} \vee \check{\alpha} = \check{\xi} \vee \check{\beta}; \quad \check{\eta} \vee \check{\alpha} = \check{\eta} \vee \check{\beta}. \quad (20.3)$$

To prove that there exists a $\{\vee, 0\}$ -homomorphism $\mu : \text{Conc } P \rightarrow D$ that satisfies the equalities

Fig. 3. The congruence lattice of P .

$$\begin{aligned} \mu(\xi) &= \varphi \Theta_L(o, a \wedge b); & \mu(\eta) &= \varphi \Theta_L(a \vee b, i); \\ \mu(\alpha) &= \varphi \Theta_L^+(a, b); & \mu(\beta) &= \varphi \Theta_L^+(b, a), \end{aligned} \quad (20.4)$$

it suffices to prove that the elements of D that lie on the right-hand sides of the four equalities in (20.4) satisfy the relations (20.3), which is an easy verification. Hence the map j is a homomorphism from $\langle K, \lambda \rangle$ to $\langle P, \mu \rangle$. \square

Notation 20.4. For $o, i, a, b, c \in L$ such that $o \leq \{a, b\} \leq i$, we define $c = a \oplus b$ to hold in $[o, i]$, if $a \wedge b = o$ and $a \vee b = c$.

Lemma 20.5. Let $o, a, b, i \in L$ such that $o \leq \{a, b\} \leq i$. Then there exist $a_0, a_1, b_0, b_1 \in L$ such that the following conditions hold:

- (i) $a = a_0 \oplus a_1$ and $b = b_0 \oplus b_1$ in $[o, i]$;
- (ii) $\Theta_L(o, a_0) = \Theta_L(o, a_1) = \Theta_L(o, a)$ and $\Theta_L(o, b_0) = \Theta_L(o, b_1) = \Theta_L(o, b)$;
- (iii) $\Theta_L(a_l \vee b_l, a \vee b) = \Theta_L(o, a \vee b)$, for all $l < 2$.

Proof. We put $K = \{o, a \wedge b, a, b, a \vee b, i\}$, and we let $f : K \hookrightarrow L$ be the inclusion map. Put $\lambda = \varphi \circ \text{Con}_c f$. As in the proofs of Lemmas 20.1 and 20.3, it suffices to find a finite partial lattice P , endowed with a $\{\vee, 0\}$ -homomorphism $\mu : \text{Con}_c P \rightarrow D$, an internal embedding $j : \langle K, \lambda \rangle \hookrightarrow \langle P, \mu \rangle$, and elements a_0, a_1, b_0, b_1 of P satisfying (i)–(iii) above in P .

We use Schmidt's well-known $M_3[K]$ construction, see [20]: namely, we put

$$P = M_3[K] = \{ \langle x, y, z \rangle \in K^3 \mid x \wedge y = x \wedge z = y \wedge z \},$$

endowed with the component-wise ordering. Since K is finite, P is a lattice. Furthermore, the canonical embedding $j : K \hookrightarrow P, x \mapsto \langle x, x, x \rangle$ is internal and congruence-preserving, see [20] or [12]. Put $\mu = \lambda \circ (\text{Con}_c j)^{-1}$. So, j is an internal embedding from $\langle K, \lambda \rangle$ into $\langle P, \mu \rangle$.

Now we put $a_0 = \langle a, o, o \rangle$, $a_1 = \langle o, a, o \rangle$, $b_0 = \langle b, o, o \rangle$, and $b_1 = \langle o, b, o \rangle$. Hence $a_0 \wedge a_1 = \langle o, o, o \rangle$ and $a_0 \vee a_1$ is the least element of P above $\langle a, a, o \rangle$, namely, $\langle a, a, a \rangle$, that is, $j(a)$. Hence $j(a) = a_0 \oplus a_1$. Similarly, $j(b) = b_0 \oplus b_1$. So (i) follows.

For $e = \langle o, o, i \rangle$, we have $a_0 \oplus e = a_1 \oplus e$ and $b_0 \oplus e = b_1 \oplus e$; thus $\Theta_P(j(o), a_0) = \Theta_P(j(o), a_1) = \Theta_P(j(o), j(a))$. Similarly, we prove $\Theta_P(j(o), b_0) = \Theta_P(j(o), b_1) = \Theta_P(j(o), j(b))$. So (ii) follows.

Finally, $a_0 \vee b_0 = \langle a \vee b, o, o \rangle$ and $a_1 \vee b_1 = \langle o, a \vee b, o \rangle$, whence

$$j(a \vee b) = (a_0 \vee b_0) \oplus (a_1 \vee b_1) \quad \text{in } P.$$

It follows that

$$\Theta_P(a_l \vee b_l, j(a \vee b)) = \Theta_P(j(o), a_{1-l} \vee b_{1-l}) = \Theta_P(j(o), j(a \vee b)),$$

for any $l < 2$, so (iii) follows. \square

Lemma 20.6. *Let $o, a, b, i \in L$ such that $o \leq \{a, b\} \leq i$. If $\varphi\Theta_L(o, a) = \varphi\Theta_L(o, b)$, then $\Theta_L(o, a) = \Theta_L(o, b)$. More precisely, there are $a_0, a_1, b_0, b_1 \in L$ such that*

- (i) $a = a_0 \oplus a_1$ and $b = b_0 \oplus b_1$ in $[o, i]$;
- (ii) a_0 and b_0 (respectively a_1 and b_1) are perspective in $[o, i]$.

Proof. Let a_0, a_1, b_0 , and b_1 be as in Lemma 20.5. By Lemma 20.5(ii), $\Theta_L(o, a_l) = \Theta_L(o, a)$ and $\Theta_L(o, b_l) = \Theta_L(o, b)$, for all $l < 2$. It follows from our assumptions that $\varphi\Theta_L(o, a_l) = \varphi\Theta_L(o, b_l)$. Furthermore,

$$\Theta_L(a_l, a \vee b) = \Theta_L(a_l, a_l \vee b_l) \vee \Theta_L(a_l \vee b_l, a \vee b) = \Theta_L(o, a \vee b),$$

for all $l < 2$, and, similarly, $\Theta_L(b_l, a \vee b) = \Theta_L(o, a \vee b)$.

It follows then from Lemma 20.3 that a_0 and b_0 (respectively a_1 and b_1) are perspective in $[o, i]$. \square

Lemma 20.7. *The map φ is an isomorphism from $\text{Con}_c L$ onto an ideal of D . If, in addition, $\langle L, \varphi \rangle$ is either lower saturated or upper saturated, then φ is an isomorphism from $\text{Con}_c L$ onto D .*

Proof. We first prove that φ is one-to-one. Let $\alpha, \beta \in \text{Con}_c L$ such that $\varphi(\alpha) = \varphi(\beta)$. By Lemma 20.1, L is relatively complemented, thus there are $o, a, b \in L$ such that $o \leq a, o \leq b, \alpha = \Theta_L(o, a)$, and $\beta = \Theta_L(o, b)$. In particular, $\varphi\Theta_L(o, a) = \varphi\Theta_L(o, b)$. By Lemma 20.6, $\alpha = \beta$.

We prove next that the range of φ is an ideal of D . Since it is a $\{\vee, 0\}$ -subsemilattice of D , it suffices to prove that the range of φ is a lower subset of D . So let α be an element of the lower subset of D generated by the range of φ , we prove that α belongs to the range of φ . There are elements $o \leq i$ of L such that $\alpha \leq \varphi\Theta_L(o, i)$. If $\alpha = 0$ or $\alpha = \varphi\Theta_L(o, i)$, then α belongs to the range of φ . Now suppose that $0 < \alpha < \varphi\Theta_L(o, i)$. Put $K = \{o, i\}$, let $f: K \hookrightarrow L$ be the inclusion map, and let $\lambda = \varphi \circ \text{Con}_c f$. Let $P = \{o, x, i\}$ be the three-element chain, with $o < x < i$, and let $j: K \hookrightarrow P$ be the inclusion map. Endow P with the $\{\vee, 0\}$ -homomorphism $\mu: \text{Con}_c P \rightarrow D$ defined by $\mu\Theta_P(o, x) = \alpha$

and $\mu\Theta_P(x, i) = \varphi\Theta_L(o, i)$. Observe that $\langle P, \mu \rangle$ is a proper D -measured partial lattice and that j is an internal embedding from $\langle K, \lambda \rangle$ into $\langle P, \mu \rangle$. Since $\langle L, \varphi \rangle$ is internally saturated, there exists a homomorphism $g: \langle P, \mu \rangle \rightarrow \langle L, \varphi \rangle$ such that $g \circ j = f$. Hence the element $\alpha = \mu\Theta_P(o, x) = (\varphi \circ \text{Conc } g)(\Theta_P(o, x))$ belongs to the range of φ .

Assume, finally, that $\langle L, \varphi \rangle$ is either lower saturated or upper saturated. Let $\alpha \in D$, we prove that α belongs to the range of φ . We do it, for example, for lower saturated $\langle L, \varphi \rangle$. The conclusion is obvious if $\alpha = 0$, so suppose that $\alpha > 0$. Pick any element o of L , and put $K = \{o\}$, endowed with the zero homomorphism from $\text{Conc } K$ to D . Let $P = \{o, x\}$ be the two-element chain, with $o < x$, endowed with the $\{\vee, 0\}$ -homomorphism $\mu: \text{Conc } P \rightarrow D$ defined by $\mu\Theta_P(o, x) = \alpha$. Then j is a lower embedding from $\langle K, \lambda \rangle$ into the proper D -measured partial lattice $\langle P, \mu \rangle$, with α in the range of μ . We conclude as in the previous paragraph that α belongs to the range of φ . \square

We record in Proposition 20.8 the information that we gathered in this section about internally saturated D -measured partial lattices.

Proposition 20.8. *Let D be a distributive lattice with zero, let $\langle L, \varphi \rangle$ be an internally saturated D -measured partial lattice. Then the following assertions hold:*

- (i) L is relatively complemented.
- (ii) The map φ is an isomorphism from $\text{Conc } L$ onto an ideal of D .
- (iii) For $o, a, b, i \in L$ such that $o \leq \{a, b\} \leq i$, $\Theta_L(o, a) = \Theta_L(o, b)$ iff there are $a_0, a_1, b_0, b_1 \in [o, i]$ such that the following conditions hold:
 - (a) $a = a_0 \oplus a_1$ and $b = b_0 \oplus b_1$ in $[o, i]$;
 - (b) a_0 and b_0 (respectively a_1 and b_1) are perspective in $[o, i]$.
- (iv) If, in addition, $\langle L, \varphi \rangle$ is either lower saturated or upper saturated, then φ is an isomorphism from $\text{Conc } L$ onto D .

Now let K , D , and φ be as in the statement of Theorem C. We replace K by the image of its kernel projection (see Lemma 18.4), so that without loss of generality, φ isolates 0. By Proposition 19.3, there exist a lower saturated D -measured partial lattice $\langle L, \alpha \rangle$ and a lower embedding $f: \langle K, \varphi \rangle \hookrightarrow \langle L, \alpha \rangle$ such that $|L| = |K| + |D| + \aleph_0$. Since $\langle L, \alpha \rangle$ is lower saturated, it follows from Proposition 20.8(iv) that φ is an isomorphism from $\text{Conc } L$ onto D . The proof is similar if “lower” is replaced by “upper.”

If the range of φ is cofinal in D , a similar argument to the one above works if we replace “lower embedding” by “internal embedding” and “lower saturated” by “internally saturated.” This completes the proof of Theorem C.

Now we can prove Theorem D. Indeed, let K , P , Q , f , g , μ , and ν satisfy the assumption of Theorem D (which is the same as the assumption of Theorem B). We first use Theorem B to find a lattice L' , homomorphisms of partial lattices $f': P \rightarrow L'$ and $g': Q \rightarrow L'$, and a $\{\vee, 0\}$ -homomorphism $\varphi': \text{Conc } L' \rightarrow D$ isolating zero such that $f' \circ f = g' \circ g$, $\mu = \varphi' \circ \text{Conc } f'$, $\nu = \varphi' \circ \text{Conc } g'$, and L' is generated, as a lattice, by $\bar{f}[P] \cup \bar{g}[Q]$. Then we apply Theorem C to $\varphi': \text{Conc } L' \rightarrow D$, to find a relatively complemented lattice L of cardinality $|L'| + |D| + \aleph_0$, a lattice homomorphism $h: L' \rightarrow L$, and an isomorphism $\varphi: \text{Conc } L \rightarrow D$ such that the following assertions hold:

- (i) $\varphi' = \varphi \circ \text{Con}_c h$.
- (ii) The range of h is cinitial (respectively cofinal) in L .
- (iii) If the range of φ' is cofinal in D , then the range of h is internal in L .

Then $\bar{f} = h \circ f'$ and $\bar{g} = h \circ g'$ satisfy the required conditions. This completes the proof of Theorem D.

21. A few consequences of Theorem C

As a special case of Theorem C (for the case where φ is an isomorphism), we obtain the following result.

Corollary 21.1. *Every lattice K such that $\text{Con}_c K$ is a lattice has an internal, congruence-preserving embedding into a relatively complemented lattice.*

The other extreme application case of Theorem C is for K being the trivial lattice and φ the zero map:

Corollary 21.2. *Let D be a distributive lattice with zero. Then there exists a relatively complemented lattice L with zero such that $\text{Con}_c L \cong D$. Furthermore, if D is bounded, then one can take L bounded.*

Actually, by using more of Theorem C, we can obtain a better representation result than Corollary 21.2:

Corollary 21.3. *Let S be a distributive $\{\vee, 0\}$ -semilattice that can be expressed as the direct limit of a countable sequence of distributive lattices with zero and $\{\vee, 0\}$ -homomorphisms. Then there exists a relatively complemented lattice L with zero such that $\text{Con}_c L \cong S$. If, in addition, S is bounded, then one can take L bounded.*

Proof. We assume that S is the direct limit of $\langle D_n \mid n < \omega \rangle$, with transition $\{\vee, 0\}$ -homomorphisms $\varphi_n : D_n \rightarrow D_{n+1}$, for $n < \omega$. If, in addition, S is bounded, then we can suppose that the D_n 's are bounded and that the φ_n 's are $\{\vee, 0, 1\}$ -homomorphisms. We construct by induction a relatively complemented lattice L_n , a lattice homomorphism $f_n : L_n \rightarrow D_{n+1}$, and an isomorphism $\alpha_n : \text{Con}_c L_n \rightarrow D_n$.

By Corollary 21.2, there exists a relatively complemented lattice L_0 with zero such that $\text{Con}_c L_0 \cong D_0$; let $\alpha_0 : \text{Con}_c L_0 \rightarrow D_0$ be any isomorphism. If D_0 has a unit, then we can suppose that L_0 is bounded.

Suppose having constructed a lattice L_n and an isomorphism $\alpha_n : \text{Con}_c L_n \rightarrow D_n$. We apply Theorem C to the $\{\vee, 0\}$ -homomorphism $\varphi_n \circ \alpha_n : \text{Con}_c L_n \rightarrow D_{n+1}$. We obtain a relatively complemented lattice L_{n+1} , a zero-preserving lattice homomorphism

$f_n : L_n \rightarrow L_{n+1}$, and an isomorphism $\alpha_{n+1} : \text{Con}_c L_{n+1} \rightarrow D_{n+1}$ such that the following diagram is commutative:

$$\begin{array}{ccc} \text{Con}_c L_n & \xrightarrow{\text{Con}_c f_n} & \text{Con}_c L_{n+1} \\ \alpha_n \downarrow & & \downarrow \alpha_{n+1} \\ D_n & \xrightarrow{\varphi_n} & D_{n+1}. \end{array}$$

Furthermore, in case S is bounded, the map $\varphi_n \circ \alpha_n$ is cofinal, so we can take f_n with internal range.

Hence the sequence $\langle L_n \mid n < \omega \rangle$ of lattices, endowed with the sequence of transition maps $f_n : L_n \rightarrow L_{n+1}$, determines a direct limit system, whose image under the Con_c functor is isomorphic, via the α_n 's, to the direct system $\langle D_n \mid n < \omega \rangle$ with the φ_n 's. Since the Con_c functor preserves direct limits, it follows from this that $\text{Con}_c L$ is isomorphic to S . In case S is bounded, all the L_n 's are bounded and all the f_n 's are $\{0, 1\}$ -embeddings, thus L is bounded. \square

22. Open problems

Let p be either a prime number or zero. We denote by \mathbf{V}_p the quasivariety of all lattices that embed into the subspace lattice of a vector space over the prime field \mathbb{F}_p of characteristic p .

Problem 1. Does every lattice in \mathbf{V}_p have a congruence-preserving relatively complemented extension in \mathbf{V}_p ?

It may be the case that a more natural context for Problem 1 is not provided by the congruence lattice, but the *dimension monoid*, see [27]. The corresponding reformulation of Problem 1 is then the following.

Problem 2. Does every lattice in \mathbf{V}_p have a dimension-preserving relatively complemented extension in \mathbf{V}_p ?

As in [27], we say that a lattice homomorphism $f : K \rightarrow L$ is *dimension preserving*, if the map $\text{Dim } f : \text{Dim } K \rightarrow \text{Dim } L$ is an isomorphism.

Problem 3. Let S be the $\{\vee, 0\}$ -direct limit of a countable sequence of distributive lattices with zero. Does there exist a relatively complemented lattice L in \mathbf{V}_p such that $\text{Con}_c L \cong S$?

If K is a sublattice of a lattice L , we say that L is an *automorphism-preserving extension* of K , if every automorphism of K extends to a unique automorphism of L and K is closed under all automorphisms of L .

Problem 4. Let K be a lattice such that $\text{Con}_c K$ is a lattice. Does K have a relatively complemented, congruence-preserving, automorphism-preserving extension?

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References

- [1] S. Burris, H.P. Sankappanavar, A Course in Universal Algebra, The millennium edition, 1999, online manuscript available at <http://www.thoralf.uwaterloo.ca>. Previously published as: Grad. Texts in Math., Vol. 78, Springer-Verlag, New York, 1981. Out of print.
- [2] R.A. Dean, Free lattices generated by partially ordered sets and preserving bounds, *Canad. J. Math.* 16 (1964) 136–148.
- [3] R. Freese, J. Ježek, J.B. Nation, Free Lattices, in: *Math. Surveys Monogr.*, Vol. 42, Amer. Math. Soc., 1995.
- [4] R. Freese, W. Lampe, W. Taylor, Congruence lattices of algebras of fixed similarity type, I, *Pacific J. Math.* 82 (1980) 59–68.
- [5] G. Grätzer, General Lattice Theory, second edition, New appendices by the author with B.A. Davey, R. Freese, B. Ganter, M. Greferath, P. Jipsen, H.A. Priestley, H. Rose, E.T. Schmidt, S.E. Schmidt, F. Wehrung, R. Wille, Birkhäuser, Basel, 1998.
- [6] G. Grätzer, H. Lakser, E.T. Schmidt, Congruence representations of join-homomorphisms of distributive lattices: a short proof, *Math. Slovaca* 46 (4) (1996) 363–369.
- [7] G. Grätzer, H. Lakser, E.T. Schmidt, Representing isotone maps as maps of congruences. II. Concrete maps, *Acta Math. Hungar.* 94 (4) (2001) 253–258.
- [8] G. Grätzer, H. Lakser, F. Wehrung, Congruence amalgamation of lattices, *Acta Sci. Math. (Szeged)* 66 (2000) 339–358.
- [9] G. Grätzer, E.T. Schmidt, Characterizations of congruence lattices of abstract algebras, *Acta Sci. Math. (Szeged)* 24 (1963) 34–59.
- [10] G. Grätzer, E.T. Schmidt, Congruence-preserving extensions of finite lattices to sectionally complemented lattices, *Proc. Amer. Math. Soc.* 127 (1999) 1903–1915.
- [11] G. Grätzer, E.T. Schmidt, Congruence Lattices, Appendix C in [5], pp. 519–530.
- [12] G. Grätzer, F. Wehrung, Proper congruence-preserving extensions of lattices, *Acta Math. Hungar.* 85 (1999) 169–179.
- [13] A.P. Huhn, On the representation of algebraic distributive lattices II, *Acta Sci. Math. (Szeged)* 53 (1989) 3–10.
- [14] A.P. Huhn, On the representation of algebraic distributive lattices III, *Acta Sci. Math. (Szeged)* 53 (1989) 11–18.
- [15] T. Jech, Set Theory, Academic Press, New York, 1978.
- [16] T. Jech, Boolean-valued models, in: J.D. Monk (Ed.), in: *Handbook of Boolean algebras*, Vol. 3, North-Holland, 1989, pp. 1197–1211.
- [17] M. Ploščica, J. Tůma, F. Wehrung, Congruence lattices of free lattices in non-distributive varieties, *Colloq. Math.* 76 (2) (1998) 269–278.
- [18] P. Pudlák, A new proof of the congruence lattice representation theorem, *Algebra Universalis* 6 (1976) 269–275.
- [19] P. Pudlák, On congruence lattices of lattices, *Algebra Universalis* 20 (1985) 96–114.
- [20] E.T. Schmidt, Zur Charakterisierung der Kongruenzverbände der Verbände, *Mat. Časopis Sloven. Akad. Vied.* 18 (1968) 3–20.

- [21] E.T. Schmidt, The ideal lattice of a distributive lattice with 0 is the congruence lattice of a lattice, *Acta Sci. Math. (Szeged)* 43 (1981) 153–168.
- [22] E.T. Schmidt, A Survey on Congruence Lattice Representations, in: *Teubner-Texte Math.*, Band 42, Teubner, Leipzig, 1982.
- [23] J. Tüma, On the existence of simultaneous representations, *Acta Sci. Math. (Szeged)* 64 (1998) 357–371.
- [24] J. Tüma, F. Wehrung, Simultaneous representations of semilattices by lattices with permutable congruences, *Internat. J. Algebra Comput.* 11 (2) (2001) 217–246.
- [25] J. Tüma, F. Wehrung, Unsolvable one-dimensional lifting problems for congruence lattices of lattices, *Forum Math.* 14 (2002) 483–493.
- [26] F. Wehrung, Boolean universes above Boolean models, *J. Symbolic Logic* 58 (4) (1993) 1219–1250.
- [27] F. Wehrung, The dimension monoid of a lattice, *Algebra Universalis* 40 (1998) 247–411.